Measure Theory: Midterm.

Oct 10, 2014

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 80 mins. The exam has a total of 4 questions and 40 points.*
- *You may use without proof* standard *results from the syllabus which are independent of the question asked,* unless *explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

[10] 1. True or false:

If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $\displaystyle\int_{\mathbb{R}} |f| \, d\lambda < \infty$, then $\displaystyle\lim_{n\to\infty} \int_{[n,2n]} f \, d\lambda$ exists.

Prove it, or find a counter example.

[10] 2. True or false:

If $f : \mathbb{R}^d \to [0,1]$ is Lebesgue measurable, then there exists $g : \mathbb{R}^d \to [0,1]$ such that g is Borel measurable, and $f = g$ almost everywhere?

Prove it, or find a counter example. [This is very similar to an optional question on HW4, and something I stated (but did not prove) in class. Please provide a complete proof here, without relying on the optional question (or what I stated but didn't prove) in class.]

[10] 3. Given a function $f : [0,1] \to [0,1]$, we define it's graph $\Gamma_f \subset \mathbb{R}^2$ by $\Gamma_f \stackrel{\text{def}}{=} \{(x, f(x)) \mid x \in [0,1]\}$. True or false:

If $f : [0,1] \to [0,1]$ is Borel measurable, then $\lambda^*(\Gamma_f) = 0$.

Prove it, or find a counter example. [Here λ^* denotes the Lebesgue outer measure on $\mathbb{R}^2$.]

[10] 4. Let $\phi_n : \mathbb{R} \to [0,1]$ be a sequence of integrable functions such that $(\phi_n) \to 0$ pointwise, and for every $n \in \mathbb{N}$ we have $\int_{\mathbb{R}} \phi_n \, d\lambda = 1$. True or false:

If $f : \mathbb{R} \to [0,1]$ is continuous, then $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}-\{0\}} \phi_n(t) f\Big(\frac{1}{t}\Big) \, dt$ exists.

Prove it, or find a counter example.