Measure Theory: Final.

Dec 12, 2014

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 8 questions and 70 points.*
- *You may use without proof* standard *results from the syllabus which are independent of the question asked,* unless *explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

[10] 1. Given $f : X \to \mathbb{R}$ be measurable, and define $F : \mathbb{R} \to [-\infty, \infty]$ by $F(x) = \mu(f < x)$.

(a) True or false: If f is measurable, then F is left continuous. Prove it, or find a counter example.

(b) True or false: If f is measurable, then F is right continuous. Prove it, or find a counter example.

[10] 2. Let μ be a finite measure on X, and $f : X \to \mathbb{R}$ be measurable. Decide whether the limits

$$\lim_{n\to\infty} \int_X e^{-n|f|}\, d\mu \quad \text{and} \quad \lim_{n\to\infty} \int_X e^{+n|f|}\, d\mu$$

necessarily exist. If yes, compute them. Prove your answer. [By convention, if a sequence approaches ∞, we say the limit exists and is ∞.]

[10] 3. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. True or false:

For any (possibly infinite) collection of balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\mathcal{A}}$ such that

$$\bigcup_{\alpha\in\mathcal{A}} B(x_\alpha, r_\alpha) \supseteq E, \quad \text{and} \quad \sup_{\alpha\in A} r_\alpha < \infty,$$

there exists a (possibly infinite) $\mathcal{A}' \subseteq \mathcal{A}$ such that the sub-collection $\{B(x_{\alpha'}, r_{\alpha'})\}_{\alpha'\in\mathcal{A}'}$ is pairwise disjoint and

$$\bigcup_{\alpha'\in\mathcal{A}'} B(x_{\alpha'}, 5r_{\alpha'}) \supseteq E.$$

Prove it, or find a counter example.

[10] 4. Let X be a compact metric space, $C(X)$ denote the set of continuous real valued functions on X. True or false:

If μ is a finite signed Borel measure on X, then $\|\mu\| = \sup\Big\{ \int_X f\, d\mu \mid f \in C(X) \text{ and } \sup_X |f| \leqslant 1 \Big\}$.

Prove it, or find a counter example. [You *may not* use the Riesz representation theorem for this question.]

[10] 5. If $f, g \in L^2(\mathbb{R}^d)$ compute $(fg)^\wedge$ in terms of $\mathcal{F}f$ and $\mathcal{F}g$. Prove it. [Recall for $f \in L^1(\mathbb{R}^d)$, we defined $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi\rangle}\, dx$ to be the Fourier transform of f, and $\mathcal{F}$ denotes the extension of the Fourier transform to L^2. Hint: First compute $(\hat{f} * \hat{g})^\wedge$ if f, g are Schwartz functions.]

6. Given $f : \mathbb{R}^2 \to \mathbb{R}$, define $G, H : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \sup_{y\in\mathbb{R}} f(x,y) \quad \text{and} \quad H(x) = \begin{cases} \operatorname*{ess\,sup}_{y\in\mathbb{R}} f(x,y) & \text{if the function } y \mapsto f(x,y) \text{ is Lebesgue measurable,} \\ 0 & \text{otherwise.} \end{cases}$$

Recall, ess sup is the essential supremum, defined by $\operatorname{ess\,sup}_y f(x,y) = \sup\{z \mid \lambda\{t \mid f(x,t) > z\} > 0\}$.

[4] (a) True or false: If f is Lebesgue measurable, then so is G. Prove it, or find a counter example.

[2] (b) True or false: If f is Borel measurable, then so is G. *No proof required!* Incorrect answers are worth no credit, blank answers half credit and correct answers full credit.

4 (c) True or false: If f is Borel measurable, then so is H. Prove it, or find a counter example.

10 7. For any $t \in [0,1]$ and $N \in \mathbb{N}$ define $\Delta_{N,t} : \mathbb{R} \to \mathbb{R}$ by

$$\Delta_{N,t}(x) = \sup\left\{t + \frac{k}{N} \;\middle|\; k \in \mathbb{Z} \text{ and } t + \frac{k}{N} \leqslant x\right\}.$$

True or false:

> If $f \in L^1(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq [0,1]$, then there exists an increasing sequence of integers $N_k \to \infty$ such that $\lim_{k \to \infty} \int_{\mathbb{R}} \left| f(x) - f\big(\Delta_{N_k,t}(x)\big) \right| dx = 0$ for *almost every* $t \in [0,1]$.

Prove it, or find a counter example. [Hint: Play with $\int_0^1 \int_{\mathbb{R}} |f(x) - f(\Delta_{N,t}(x))| \, dx \, dt$.]

If you've completed the remainder of this exam and have time to spare, here is a fun question. This is for your entertainment only, and *will not influence your grade.*

0 8. Let X be a locally compact metric space, and μ a regular Borel measure on X. Suppose $\mu(\{x\}) = 0$ for every $x \in X$. If $F \in \mathcal{B}(x)$ has finite measure, and $0 < \alpha < \mu(F)$, show that there exists $A \in \mathcal{B}(X)$ such that $A \subset F$ and $\mu(A) = \alpha$.