Stochastic Integration.

Fix $M \in \mathcal{M}_c^2$. Want to define the Itô integral $I_t(X) = \int_0^t X_s\,dM_s$

Can't blindly do Riemann Sums: $\lim \sum X_{\xi_i}(M_{t_{i+1}} - M_{t_i})$. $t \to M_t$ is not BV/a.e.

so all not converge. Itô: If X is adapted & $\xi_i = t_i$ then CAN do the above!!

Def: X is a simple process if it is adapted, bounded, LCRL and piecewise constant in time.

i.e. $\exists\, t_0 = 0 < t_1 < t_2 \cdots$ s.t. $(t_n) \to \infty$ & a family of random variables $\{\xi_i\}$

s.t. $\{\xi_i\}$ is uniformly bdd, $\forall i$ ξ_i is $\mathcal{F}_{t_i}$ meas & $X_t = \begin{cases} \xi_0 & t = 0 \\ \xi_i & t \in (t_i, t_{i+1}] \end{cases}$

If X is as above, we define $I_t(X) = \sum X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})$. Explicitly if

$t \in [t_n, t_{n+1})$, $I_t(X) = \sum_0^{n-1} X_{t_i}(M_{t_{i+1}} - M_{t_i}) + X_{t_n}(M_t - M_{t_n})$

Prop $I(X) \in \mathcal{M}_c^2$ and $\langle I(X)\rangle_t = \int_0^t X_s^2\, d\langle M\rangle_s$. ($\langle M\rangle$ B.V. RHS Stieltjes integral)

Pf: Continuity in time is immediate. For $s < t$, consider three cases.

Notation $E_s = E(\cdot \mid \mathcal{F}_s)$ (1) $t_i \le s$. Then $E_s X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}) = X_{t_i} E_s(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}) = X_{t_i}(M_{t_{i+1}\wedge s} - M_{t_i \wedge s})$

(2) $t > t_i > s$. Then $E_s X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}) = E_s E_{t_i}(X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})) = E_s(X_{t_i} 0)$

$= X_{t_i}(M_{t_{i+1}\wedge s} - M_{t_i\wedge s})$. (3) $s \le t_i \Rightarrow E_{t_i}(-)_t = 0 = E_{t_i}(-)_s$. $\therefore$ (1)+(2)+(3) $\Rightarrow I(X) \in \mathcal{M}_c$.

For $\langle I(X)\rangle$, NTS $E_s\left(I_t(X)^2 - \int_0^t X_r^2 \langle M\rangle_r\right) = I_s(X)^2 - \int_0^s X_r^2\, d\langle M\rangle_r$

i.e. NTS $E_s(I_t(X)^2 - I_s(X)^2) = E_s\left(\int_s^t X_r^2\, d\langle M\rangle_r\right)$. For this,

$E_s(I_t(X)^2 - I_s(X)^2) = E_s((I_t(X) - I_s(X))^2) = \longrightarrow$ (when $s \in [t_k, t_{k+1})$)

$= E_s\left[\left(\sum_{i \ge k+1} X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}) + X_{t_k}\left[(M_{t_{k+1}\wedge t} - \cancel{M_{t_k}}) - (M_s - \cancel{M_{t_k}})\right]\right)^2\right]$

(1) All cross terms are 0: $E_s(X_{t_i}X_{t_j}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})(M_{t_{j+1}\wedge t} - M_{t_j\wedge t}))$

$(s \le t_i < t_j)$ $= E_s\left[(\quad) E_{t_j}(M_{t_{j+1}\wedge t} - M_{t_j \wedge t})\right] = 0$

(2) $E_s(X_{t_i}^2 (M^2_{t_{i+1}\wedge t} - M^2_{t_i \wedge t})) = E_s(X_{t_i}^2 E_{t_i}(M^2_{t_{i+1}\wedge t} - M^2_{t_i\wedge t}))$ $\Leftarrow$ ($\because M^2 - \langle M\rangle \in \mathcal{M}_c$)

$= E_s(X_{t_i}^2(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i \wedge t}))$ $= E_s\left(X_{t_i}^2 \int_{t_i\wedge t}^{t_{i+1}\wedge t} d\langle M\rangle\right) = E_s\left(\int_{t_i \wedge t}^{t_{i+1}\wedge t} X_r^2\, d\langle M\rangle_r\right)$

& so $E(I_t(X)^2 - I_s(X)^2 \mid \mathcal{F}_s) = E\left(\int_s^t X_r^2\, d\langle M\rangle_r \,\middle|\, \mathcal{F}_s\right)$ QED

Cor: (Itô isometry) norm in $\mathcal{M}_c^2[0,T] \longrightarrow E\left(\int_0^T X_s\, dM_s\right)^2 = E\int_0^T X_s^2\, d\langle M\rangle_s \leftarrow$ norm in $L^2(\Omega \times [0,T], P \otimes d\langle M\rangle)$

Notation: $\mathcal{L}_0 = \{\text{simple processes}\}$. $\mathcal{L} = \{\text{meas, adapted processes } X \mid \forall t\ E\int_0^t X_s^2\, d\langle M\rangle_s < \infty\}$

$\mathcal{L}^* = \{\text{prog meas processes } X \mid \forall t,\ E\int_0^t X_s^2\, d\langle M\rangle_s < \infty\}$.

For $X \in \mathcal{L}$, let $\|X\|_{\mathcal{L}} = \sum \frac{1}{2^n}\left(E\int_0^n X_s^2\, d\langle M\rangle_s\right) \wedge 1$

Lemma: $\mathcal{L}_0$ is dense in $\mathcal{L}^*$. Also, if $t \to \langle M\rangle_t$ is a.c. then $\mathcal{L}_0$ is dense in $\mathcal{L}$ (Pf later).

Construction of the Itô integral: By Itô isometry, $\forall X \in \mathcal{L}_0$, $\|X\|_{\mathcal{L}} = \|I(X)\|_{\mathcal{M}_c^2}$

Now $\forall X \in \mathcal{L}^*$, let $(X^{(n)})$ be a sequence in $\mathcal{L}_0$ s.t. $(X^{(n)}) \xrightarrow{\mathcal{L}^*} X$

$\Rightarrow (X^{(n)})$ is Cauchy in $\mathcal{L}^*$ & by Itô isom $\Rightarrow (I(X^{(n)}))$ is Cauchy in $\mathcal{M}_c^2$.

Define $I(X) = \lim_{n\to\infty} I(X^{(n)}) \in \mathcal{M}_c^2$ [Note: $I(X)$ is independent of the choice of subsequence $(X^{(n)})$.] Similarly define $I(X)$ $\forall X \in \mathcal{L}$, if $\langle M\rangle$ is a.c.

Rmk If $X \in \mathcal{L}$ is continuous in time, then $\forall t$, $\int_0^t X_s\, dM_s = \lim_{|\Delta| \to 0} \sum_{t_i \in \Delta} X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})$

which is like a Stieltjes integral, <u>sampled at the left endpoint.</u>

Prop: Linearity: $I(X + \alpha Y) = I(X) + \alpha I(Y)$ $\forall X, Y \in \mathcal{L}^*$

Prop: $\forall X, Y \in \mathcal{L}^*$, $I(X), I(Y) \in \mathcal{M}_c^2$ & $\langle I(X)\rangle_t = \int_0^t X_s^2\, d\langle M\rangle_s$ & $\langle X, Y\rangle_t = \int_0^t X_s Y_s\, d\langle M\rangle_s$.

Pf: Let $(X^{(n)}) \xrightarrow{\mathcal{L}^*} X$ & $X^{(n)} \in \mathcal{L}_0$ $\forall n$. Then $I(X^{(n)}) \xrightarrow{\mathcal{M}_c^2} I(X) \Rightarrow \forall t$,

$I_t(X^{(n)})^2 \xrightarrow{L^1(\Omega)} I_t(X)^2$. Also, $\int_0^t (X_s^{(n)})^2\, d\langle M\rangle_s \xrightarrow{L^1(\Omega)} \int_0^t X_s^2\, d\langle M\rangle_s$

since $(X^{(n)}) \xrightarrow{\mathcal{L}^*} X$. Hence, $\left(I_t(X^{(n)})^2 - \int_0^t (X_s^{(n)})^2\, d\langle M\rangle_s\right) \xrightarrow{L^1(\Omega)} I_t(X)^2 - \int_0^t X_s^2\, d\langle M\rangle_s$

$\forall t$. $\Rightarrow \{I_t(X)^2 - \int_0^t X_s^2\, d\langle M\rangle_s\} \in \mathcal{M}_c \Rightarrow \langle I(X)\rangle_t = \int_0^t X_s^2\, d\langle M\rangle_s$.

Similarly, $\{I_t(X)I_t(Y) - \int_0^t X_s Y_s\, d\langle M\rangle_s\} \in \mathcal{M}_c \Rightarrow \langle I(X), I(Y)\rangle_t = \int_0^t X_s Y_s\, d\langle M, M\rangle_s$.

Prop: If $M \in \mathcal{M}^2_{c}$, $\langle M\rangle$ is a.c. (as a fn of time), then $\mathcal{L}_0$ is dense in $\mathcal{L} = \mathcal{L}(M)$

Remk: If $d\langle M\rangle$ is not a.c. then $\mathcal{L}_0$ is only dense in $\mathcal{L}^*$. (Pf in K.S.)

Lemma: Say $d\langle M\rangle = dt$ & $X \in \mathcal{L}$ is bdd. Then $\exists X^{(n)} \in \mathcal{L}_0 \ni (X^{(n)}) \xrightarrow{\mathcal{L}}$ & $\|X^{(n)}\|_{L^\infty} \le \|X\|_{L^\infty}$

Pf: Fix $T>0$. Enough to prove $\exists X^{(n)}\in\mathcal{L}_0 \ni \lim_{n\to\infty} E\int_0^T |X_t - X_t^{(n)}|^2 dt = 0$.

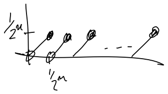
Let $\varphi_n(t) = \frac{k}{2^n}$ if $t \in (\frac{k}{2^n}, \frac{k+1}{2^n}]$. Set $X_t = 0$ $\forall t \le 0$ for simplicity

Note If X is bdd cts then can set $X^{(n)}_t = X_{\varphi_n(t)}$. If $X\in\mathcal{L}^*$, then can approximate X by averaging in time. Since neither are assumed, will show $\exists s \in \mathbb{Q} \ni X_{\varphi_n(t-s)+s} \xrightarrow{\mathcal{L}} X$ along a subsequence (i.e. sampling X at some diadic translate of $\mathbb{Q}$ works).

Note. $\forall t,s$, $\varphi_n(t-s) \in (t-s-\frac{1}{2^n}, t-s] \Rightarrow \varphi_n(t-s)+s \in (t-\frac{1}{2^n}, t]$

Also for fixed s, $X^{(n,s)}_t = X_{\varphi_n(t-s)+s}$ is simple. Then,

$$E\int_{t=0}^T\int_{s=0}^1 |X_t - X_{\varphi_n(t-s)+s}|^2 ds\,dt = E\int_{t=0}^T \int_{h=t-1}^{t} |X_t - X_{t-h+\varphi_n(h)}|^2 dh\,dt \quad (\text{Put } s=t-h)$$

$$= 2^n E\int_0^T\int_{h'=0}^{1/2^n} |X_t - X_{t-h'}|^2 dh'\,dt$$ (Since $h-\varphi_n(h) =$ )

$$\le \sup_{|h|\le \frac{1}{2^n}} E\int_0^T |X_t - X_{t-h}|^2 dt \longrightarrow 0$$ by continuity of translation on $L^2([0,T], L^2(\Omega))$

$$\Rightarrow \lim_{n\to\infty}\int_{s=0}^1 \left(E\int_{t=0}^T |X_t - X_t^{(n,s)}|^2 dt\right) = 0 \Rightarrow \exists \text{ a subsequence } n_k \ni \text{ for almost every } s\in[0,1],$$

$$\lim_{n\to\infty} E\int_{t=0}^T |X_t - X_t^{(n,s)}|^2 dt = 0$$ QED.

Pf of Prop: For $A \in \mathcal{F}\otimes\mathcal{B}[0,T]$, let $\mu = P\otimes d\langle M\rangle$ (i.e. $\mu(A) = \int \chi_A(\omega,t)\, d\langle M\rangle_t(\omega)\, dP(\omega)$) and $\lambda = P\otimes dt$ (i.e. $\lambda(A) = \int \chi_A(\omega,t)\,dt\,dP(\omega)$). $d\langle M\rangle_t \ll dt \Rightarrow \mu \ll \lambda$.

① Fix $T>0$, & Say X is bdd. Lemma $\Rightarrow \exists X^{(n)}\in\mathcal{L}_0 \ni \lim E\int_0^t |X^{(n)} - X|^2 dt = 0$.

$\Rightarrow \exists$ a subsequence $X^{(n_k)}$ & $N \subseteq \Omega\times[0,T] \ni \lambda(N) = 0$ & $(X^{(n_k)}) \to X$ on N^c

$\Rightarrow \mu(N) = 0$. Also, $|X_t - X_t^{(n_k)}|^2 \frac{d\langle M\rangle}{dt} \le 2\|X\|^2_{L^\infty}\frac{d\langle M\rangle}{dt} \in L^1(\Omega\times[0,T])$

$\Rightarrow E\int_0^T |X_t - X_t^{(n_k)}|^2 d\langle M\rangle_t \to 0 \Rightarrow \|X - X^{(n_k)}\|_{\mathcal{L}(M)} \to 0$

② If X is not bdd, set $X^{(n)} = X^+\wedge n - X^-\wedge n$. Then $|X - X^{(n)}| \le 2|X|$ & by dominated convergence $\|X^{(n)} - X\|_{\mathcal{L}(M)} \to 0$. Now find $\tilde{X}^{(n)} \in \mathcal{L}_0 \ni \|\tilde{X}^{(n)} - X^{(n)}\|_{\mathcal{L}(M)} \le \frac{1}{n}$ by ①. Then $(\tilde{X}^{(n)}) \xrightarrow{\mathcal{L}(M)} X$. QED.

Remk: Subsequently assume $d\langle M\rangle$ is a.c. (a.s). Note $d\langle I(X,M)\rangle \ll d\langle M\rangle \Rightarrow d\langle I(X,M)\rangle \ll dt$.

Prop: If $M,N\in\mathcal{M}^2_c$, $X\in\mathcal{L}(M)$, then $\langle I(X), N\rangle_t = \int_0^t X_s\, d\langle M,N\rangle_s$.

Pf: Let $(X^{(n)}) \xrightarrow{\mathcal{L}(M)} X$ be a seq of simple processes.

Then $E|\langle I(X^{(n)}),N\rangle_t - \langle I(X),N\rangle_t| = E|\langle I(X^{(n)}-X),N\rangle_t| \le \left(E\langle I(X^{(n)}-X)\rangle_t\right)^{1/2}\left(E\langle N\rangle_t\right)^{1/2}$

$$= \left(E\int_0^t |X^{(n)}_s - X_s|^2 d\langle M\rangle_s\right)^{1/2}\left(EN_t^2\right)^{1/2} \longrightarrow 0$$

Will show $E\left|\langle I(X^{(n)}),N\rangle - \int_0^t X_s\, d\langle M,N\rangle_s\right| \longrightarrow 0$ ($\Rightarrow$ QED).

Say $I_t(X^{(n)}) = \sum X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})$

$\Rightarrow \langle I(X^{(n)}), N\rangle = \sum X^{(n)}_{t_i}\left(\langle M,N\rangle_{t_{i+1}\wedge t} - \langle M,N\rangle_{t_i\wedge t}\right) = \int_0^t X^{(n)}_s\, d\langle M,N\rangle_s$ (∵ $X^{(n)}$ is simple.)

$\Rightarrow E\left|\langle I(X^{(n)}),N\rangle_t - \int_0^t X_s\, d\langle M,N\rangle_s\right| = E\left|\int_0^t (X^{(n)}_s - X_s)\, d\langle M,N\rangle_s\right|$

$$\overset{(KW)}{\le} E\left(\int_0^t |X^{(n)}_s - X_s|^2 d\langle M\rangle_s\right)^{1/2}\left(\int_0^t d\langle N\rangle_s\right)^{1/2} \le \left(E\int_0^t |X_s^{(n)}-X_s|^2 d\langle M\rangle_s\right)^{1/2}\left(EN_t^2\right)^{1/2} \longrightarrow 0$$ QED

Cor: If $M,N\in\mathcal{M}^2_c$, $X\in\mathcal{L}(M)$, $Y\in\mathcal{L}(N)$ Then $\langle I(X,M), I(Y,N)\rangle_t = \int_0^t X_sY_s\, d\langle M,N\rangle_s$

Pf: Since $I(Y,N)\in\mathcal{M}^2_c$, $\langle I(X,N), I(Y,N)\rangle_t = \int_0^t X_s\, d\langle M, I(Y,N)\rangle_s = \int_0^t X_sY_s\, d\langle M,N\rangle_s$ QED

Prop: Let $M\in\mathcal{M}^2_c$, $X\in\mathcal{L}(M)$. Then $I(X)$ is the unique element in $\mathcal{M}^2_c \ni \langle I(X),N\rangle_t = \int_0^t X_s\, d\langle M,N\rangle_s$ $\forall N\in\mathcal{M}^2_c$.

Pf: Already shown that $\langle I(X),N\rangle_t = \int_0^t X_s\, d\langle M,N\rangle_s$. Conversely, say $Y\in\mathcal{M}^2_c$ satisfies $\langle Y,N\rangle_t = \int_0^t X_s\, d\langle M,N\rangle_s$. $\Rightarrow \langle Y - I(X), N\rangle = 0$ $\forall N\in\mathcal{M}^2_c$. $\Rightarrow Y = I(X)$ in $\mathcal{M}^2_c$. QED.

Cor: Let $M\in\mathcal{M}^2_c$, $X\in\mathcal{L}^*(M)$ & set $N_t = \int_0^t X_s\, dM_s \in \mathcal{M}^2_c$. Say $Y\in\mathcal{L}^*(N)$. Then $\int_0^t Y_s\, dN_s = \int_0^t X_sY_s\, dM_s$.

(i.e. if "$dN = X\,dM$", then "$Y\,dN = XY\,dM$".)

Pf: $\forall \varphi \in \mathcal{M}^2_c$, $\langle I(Y,N),\varphi\rangle_t = \int_0^t Y_s\, d\langle N,\varphi\rangle_s = \int_0^t X_sY_s\, d\langle M,\varphi\rangle_s = \langle I(XY,M),\varphi\rangle_t$ QED.

Integration wrt continuous local martingales. Let $M \in \mathcal{M}_{c,loc}$. (Assume $\langle M\rangle$ is a.e.). Define

$\mathcal{P}(M) = \mathcal{P} = \{X \mid X$ is meas. adapted & $\int_0^T X_s^2\, d\langle M\rangle_s < \infty$ a.s. $\forall T\} \supseteq \mathcal{L}(M)$

$\mathcal{P}^*(M) = \mathcal{P}^* = \{X \mid X$ is prog meas & $\int_0^T X_s^2\, d\langle M\rangle_s < \infty$ a.s. $\forall T\}$. $\supseteq \mathcal{L}^*(M)$

Let $\tau_n = \inf\{t \mid \int_0^t X_s^2\, d\langle M\rangle_s \ge n\} \wedge n$. Note $\tau_n \le \tau_{n+1}$ & $(\tau_n) \to \infty$ a.s.

τ_n' be a localizing seq for M, & let $T_n = \tau_n \wedge \tau_n'$. (also a bdd loc seq for M).

Let $M^{(n)} = M^{T_n} \in \mathcal{M}_c^2$ & $X^{(n)} = X^{T_n} \in \mathcal{L}(M^{(n)})$. Note for $m \le n$,

$$\int_0^{t\wedge T_m} X_s^{(n)}\, dM_s^{(n)} = \int_0^t X_s^{(n)}\, dM^{(n)}_{s\wedge T_m} = \int_0^t X_s^{(n)}\, dM_s^{(m)} = \int_0^t X_s^{(m)}\, dM_s^{(m)} \quad \text{(by HW)}$$

& so $I(X^{(m)}, M^{(m)}) = I(X^{(n)}, M^{(n)})^{T_m}$ Def: Define $I(X,M) = \lim_{n\to\infty} I(X^{(n)}, M^{(n)})$.

[Note $\forall t$, $\lim_{n\to\infty} I_t(X^{(n)}, M^{(n)})$ is eventually const. Hence $I(X,M)^{T_m} = I(X^{(m)}, M^{(m)})$ $\forall m$]

Prop: $I(X) \in \mathcal{M}_{c,loc}$ & is the unique element s.t. $\langle I(X), N\rangle_t = \int_0^t X_s\, d\langle M,N\rangle_s$ a.s. $\forall N \in \mathcal{M}_{c,loc}$

Pf: Let T_n be as in the def of I. Then $I(X)^{T_n} = I(X^{(n)}, M^{(n)}) \in \mathcal{M}_c^2 \Rightarrow I(X) \in \mathcal{M}_{c,loc}$.

Let $\tilde\tau_n$ be a loc seq for N, $\sigma_n = T_n \wedge \tilde\tau_n$, $\langle I(X), N\rangle_t^{\sigma_n} = \langle I(X)^{\sigma_n}, N^{\sigma_n}\rangle_t$

$= \langle I(X^{(n)}, M^{(n)})^{\sigma_n}, N^{\sigma_n}\rangle_t = \int_0^{t\wedge\sigma_n} X_s\, d\langle M,N\rangle_s$. Other direction is similar. QED

Def: We say $X = \{X_t, \mathcal{F}_t \mid 0 \le t < \infty\}$ is a (continuous) local semimartingale if

$X = X_0 + M + B$, where $M \in \mathcal{M}_{c,loc}$ & B is continuous, adapted, B.V. with $B_0 = 0$.

Remark: The decomposition $X = X_0 + M + B$ is unique. (Pf: If $X = X_0 + M' + B'$, then

$\mathcal{M}_{c,loc} \ni M - M' = B - B' \in BV \Rightarrow M - M' \equiv 0$.)

Prop: If $X = X_0 + M + B$ as above, then $\langle X\rangle = \langle M\rangle$. Also, if $Y = Y_0 + N + C$, $\langle X, Y\rangle = \langle M, N\rangle$.

Pf: Let $\Delta: t_0 = 0 < t_1 < \dots < t_n = T$. Then $\langle X\rangle_t^\Delta = \sum (X_{t_{i+1}\wedge t} - X_{t_i\wedge t})^2 = \sum (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2$

$$+ 2\sum (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})(B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) + \sum (B_{t_{i+1}\wedge t} - B_{t_i\wedge t})^2$$

$$\Rightarrow |\langle X\rangle_t^\Delta - \langle M\rangle_t^\Delta| \le \underbrace{\langle B\rangle_t^\Delta}_{\to 0} + 2\underbrace{\sqrt{\langle M\rangle_t^\Delta}}_{\to \langle M\rangle^\Delta}\underbrace{\sqrt{\langle B\rangle_t^\Delta}}_{\to 0} \xrightarrow{P} 0 \quad \text{(unif in time)} \quad \text{QED}$$

Let $Y \in \mathcal{P}^*(M)$ be such that for a.e. $\omega \in \Omega$, $Y \in L^1_{loc}(dB(\omega))$. Then we define

$$\int_0^t Y_s\, dX_s = \overbrace{\int_0^t Y_s\, dM_s}^{\text{Itô}} + \int_0^t Y_s\, dB_s \leftarrow \text{Stieltjes}$$

Thm (1D Itô) Let X be a continuous 1D local semimartingale, $f \in C^2(\mathbb{R})$. Then (a.s.)

$$f(X_T) - f(X_0) = \int_0^T f'(X_s)\, dX_s + \frac12 \int_0^T f''(X_s)\, d\langle X\rangle_s \quad \left(\text{Short hand notation } df(X_t) = f'(X_t)\,dX_t + \frac12 f''(X_t)\, d\langle X\rangle_t\right)$$

Notation: $f(X_T) - f(X_0) = \int_0^T \text{"}df(X)\text{"} = \int_0^T \left(f'(X)\,dX + \frac12 f''(X_s)\, d\langle X\rangle_s\right)$, $\forall T$. Hence we the short hand.

Remark: For all $\omega \in \Omega$, & fixed $t < \infty$, $\{f'(X_s(\omega)) \mid s \le t\}$ is bdd $\Rightarrow f'(X) \in \mathcal{P}^*(M) \Rightarrow \int_0^t f'(X_s)\, dX_s$ makes sense.

Thm (Multi D Itô): $f: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is $C^{1,2}$, then

$$df_t(X_t) = \partial_t f_t(X_t)\, dt + \sum_i \partial_i f_t(X_t)\, dX_t^{(i)} + \frac12 \sum_{i,j} \partial_{ij}^2 f_t(X_t)\, d\langle X^{(i)}, X^{(j)}\rangle$$

i.e. $$f(X_T) - f(X_0) = \int_0^T \partial_t f_t(X_t)\, dt + \sum_i \int_0^T \partial_i f_t(X_t)\, dX_t^{(i)} + \frac12 \sum_{i,j} \int_0^T \partial_{ij}^2 f_t(X_t)\, d\langle X^{(i)}, X^{(j)}\rangle_t$$

Thm (Levy) $M \in \mathcal{M}_{c,loc}$ (d dimensional) & $\langle M\rangle_t = tI$ (i.e. $\langle M^{(i)}, M^{(j)}\rangle_t = \delta_{ij} t$) $\Rightarrow M$ is a d-dim B.M.

Pf: Claim: $\forall \xi \in \mathbb{R}^d$, $X_t = \exp(i\langle \xi, M_t\rangle + \frac12 |\xi|^2 t)$ is a loc mg

Pf: Fix $\xi \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$, $t \ge 0$ let $f(x,t) = e^{i\langle x,\xi\rangle + \frac12|\xi|^2 t}$. Then $X_t = f(M_t, t)$.

$$\Rightarrow dX_t = \partial_t f\, dt + \sum \partial_j f_t(M)\, dM_t^{(j)} + \frac12 \sum \partial^2_{jk} f_t(M)\, d\langle M^{(j)}, M^{(k)}\rangle_t$$

$$= \cancel{\frac12 |\xi|^2 f_t(M_t)\, dt} + \sum i\xi_j f_t(M_t)\, dM^{(j)} + \frac12 \cancel{\underbrace{\Delta f(M)}_{-\frac12|\xi|^2 f(M)}\, dt} = i\sum \xi_j f_t(M_t)\, dM^{(j)}$$

$\Rightarrow X_t - X_0 = i \sum_j \xi_j \int_0^t f_s(M_s)\, dM_s^{(j)} \Rightarrow X_t - X_0 \in \mathcal{M}_{c,loc}$.

Also, $E \sup_{s \le t} |X_s| \le e^{\frac12 |\xi|^2 t} \Rightarrow X - X_0 \in \mathcal{M}_{c,loc}$ QED.

Consequently $E_s X_t = X_s \Rightarrow E_s\left(\exp(i\langle \xi, M_t\rangle + \frac12|\xi|^2 t)\right) = \exp(i\langle \xi, M_s\rangle + \frac12 |\xi|^2 s)$

$\Rightarrow E_s \exp(i\langle \xi, M_t - M_s\rangle) = e^{-\frac12|\xi|^2(t-s)} \Rightarrow M_t - M_s \perp \mathcal{F}_s$ & $M_t - M_s \sim N(0, (t-s) I_d)$ QED

Remark: If $E_s(\exp(i\langle X, \xi\rangle)) = E\exp(i\langle X, \xi\rangle)$ $\forall \xi$ then $X \perp \mathcal{F}_s$.

Pf: Let Y be $\mathcal{F}_s$ meas. $E\exp(i\langle X+Y, \xi\rangle) = EE(\ \mid \mathcal{F}_s) = E\exp(i\langle Y, \xi\rangle)\, E\exp(i\langle X, \xi\rangle)$ QED.

Recall: $f \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$; X a cts, d-dim local semi martingale, then (Note semi mg $\Rightarrow$ adapted).

$$df(X_t) = \partial_t f(X_t)\,dt + \sum \partial_i f(X_t)\,dX^{(i)} + \frac{1}{2}\sum \partial_i\partial_j f(X_t)\,d\langle X^{(i)}, X^{(j)}\rangle.$$

Pf of Itô: 1D version. X a cts 1D local semi-mg, $f \in C^2(\mathbb{R})$. $df(X_t) = f'(X_t)\,dX_t + \frac{1}{2}f''(X)\,d\langle X\rangle_t$.

Intuition: Let $\{0=t_0<t_1 \dots t_n=T\}$ be a partition of $[0,T]$. Then,

$$f(X_T)-f(X_0) = \sum f(X_{t_{i+1}}) - f(X_{t_i}) = \underbrace{\sum f'(X_{t_i})(X_{t_{i+1}}-X_{t_i})}_{\int_0^T X_s\,dX_s} + \underbrace{\frac{1}{2}f''(X_{\eta_i})(X_{t_{i+1}}-X_{t_i})^2}_{\int_0^T f''(X_s)\,d\langle X\rangle_s}.$$

Lemma: Let $X = X_0 + M + B$, $M \in \mathcal{M}_c^2$, B B.V with $B_0 = 0$. Let $\tilde{B}_t = \int_0^t |dB_s|$ & $g \in C_b(\mathbb{R})$.

Say $\exists\, C \ni \sup_{t\le T}|M_t| \le C$, $\langle M\rangle_T \le C$, $\tilde{B}_T \le C$ & $\|g\|_{L^\infty} \le C$. Then for any partition $\Delta = \{0 = t_0 < t_1 \dots t_n = T\}$, define $G^\Delta = \sum g(X_{\eta_i})(X_{t_{i+1}}-X_{t_i})^2$, where $\eta_i \in [t_i, t_{i+1})$ are arbitrary. Then if (Δ_n) is a seq of partitions with $|\Delta_n| \to 0$, $(G^{\Delta_n}) \xrightarrow[\text{along a subseq}]{a.s.} \int_0^T g(X_s)\,d\langle X\rangle_s$.

Pf: Note first, $\sum |g(X_{\eta_i}) - g(X_{t_i})|(X_{t_{i+1}}-X_{t_i})^2 \xrightarrow[|\Delta_n|\to 0]{a.s.} 0$ & is bdd above by $C\sum(X_{t_{i+1}}-X_{t_i})^2$.

But $E\,C\sum(X_{t_{i+1}}-X_{t_i})^2 \le 2C\,E\sum(M_{t_{i+1}}^2 - M_{t_i}^2) + (B_{t_{i+1}}-B_{t_i})^2 \le 2C\,M_T^2 + 2C^2 < \infty$.

$\therefore \sum |g(X_{\eta_i}) - g(X_{t_i})|(X_{t_{i+1}}-X_{t_i})^2 \xrightarrow[|\Delta_n|\to 0]{L^1(\Omega)} 0$ & it suffices to prove the lemma assuming $\eta_i = t_i$.

Also, $\sum g(X_{t_i})(B_{t_{i+1}}-B_{t_i})^2 \le \|g\|_\infty \left(\max|B_{t_{i+1}}-B_{t_i}|\right)\sum|B_{t_{i+1}}-B_{t_i}| \xrightarrow{a.s.} 0$

$\|^{ly}$, $\sum g(X_{t_i})(B_{t_{i+1}}-B_{t_i})(M_{t_{i+1}}-M_{t_i}) \le \|g\|_\infty \left(\sum(M_{t_{i+1}}-M_{t_i})^2\right)^{1/2}\left(\sum(B_{t_{i+1}}-B_{t_i})^2\right)^{1/2} \xrightarrow[\text{along a subseq}]{a.s.} 0$

So to prove the claim, it suffices to show $\sum g(X_{t_i})(M_{t_{i+1}}-M_{t_i})^2 \xrightarrow{a.s.} \int_0^T g(X_s)\,d\langle M\rangle_s$.

Note $E[(M_{t_{i+1}}-M_{t_i})^2 - (\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i})] = 0$. Hence, using adaptedness & conditioning,

Cross terms vanish.

$$E\left(\sum g(X_{t_i})[(M_{t_{i+1}}-M_{t_i})^2 - (\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})]\right)^2 = E\sum g(X_{t_i})^2\,[(M_{t_{i+1}}-M_{t_i})^2 - (\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})]^2$$

$$\le 2C^2\Big(\underbrace{E\sum(M_{t_{i+1}}-M_{t_i})^4}_{\to 0\ (4^{th}\text{ var of a mg})} + \underbrace{E\sum(\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})^2}_{\to 0\ (2^{nd}\text{ var of B.V.})}\Big)$$

$$\Rightarrow E\left[\sum g(X_{t_i})(M_{t_{i+1}}-M_{t_i})^2 - \sum g(X_{t_i})(\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})\right]^2 \to 0$$

But $\sum g(X_{t_i})(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}) \xrightarrow{a.s.} \int_0^T g(X_s)\,d\langle M\rangle_s$ $\Rightarrow$ QED.

Pf of Itô: Let $\tau_n = n \wedge \inf\{t \mid |M_t| \vee \langle M\rangle_t \vee \tilde{B}_t \vee |f(X_t)| \vee |f'(X_t)| \vee |f''(X_t)| > n\}$

Then for any partition $\Delta = \{0 = t_0 < t_1 \dots t_k = T\}$, we have

$$f(X_T^{\tau_n}) - f(X_0^{\tau_n}) = \sum f'(X_{t_i}^{\tau_n})(M_{t_{i+1}}^{\tau_n} - M_{t_i}^{\tau_n}) + f'(X_{t_i}^{\tau_n})(B_{t_{i+1}}^{\tau_n}-B_{t_i}^{\tau_n}) + \frac{1}{2}f''(X_{\eta_i}^{\tau_n})(X_{t_{i+1}}^{\tau_n}-X_{t_i}^{\tau_n})^2.$$

Lemma $\Rightarrow \forall n$, $\frac{1}{2}\sum f''(X_{\eta_i}^{\tau_n})(X_{t_{i+1}}-X_{t_i})^2 \xrightarrow[|\Delta|\to 0]{a.s.} \int_0^T f''(X_s^{(\tau_n)})\,d\langle X^{\tau_n}\rangle_s$ along a subsequence.

By Stieltjes, $\sum f'(X_{t_i}^{\tau_n})(B_{t_{i+1}}^{\tau_n} - B_{t_i}^{\tau_n}) \xrightarrow[|\Delta|\to 0]{a.s.} \int_0^t f'(X_s)\,dB_s$

Finally, set $Y_t^\Delta = f'(X_{t_i}^{\tau_n})$ for $t \in (t_i, t_{i+1}]$. Then, $E\int_0^T |Y_t^\Delta - f'(X_t^{\tau_n})|^2\,d\langle M^{\tau_n}\rangle \xrightarrow{|\Delta|\to 0} 0$

(since $Y^\Delta \to f'(X^{\tau_n})$ a.s. & $|Y^\Delta - f'(X^{\tau_n})| \le 2n$). $\therefore I_T(Y^\Delta, M^{\tau_n}) \xrightarrow[|\Delta|\to 0]{L^2(\Omega,\mathcal{F}_T)} I(f'(X^{\tau_n}), M^{\tau_n})$

$\Rightarrow \sum f'(X_{t_i}^{\tau_n})(M_{t_{i+1}}^{\tau_n} - M_{t_i}^{\tau_n}) \xrightarrow[|\Delta|\to 0]{a.s.} \int_0^T f'(X_s^{\tau_n})\,dM_s^{\tau_n}$ along a subsequence.

$\therefore f(X_T^{\tau_n}) - f(X_0^{\tau_n}) = \int_0^T f'(X_s^{\tau_n})\,dX_s^{\tau_n} + \frac{1}{2}\int_0^T f''(X_s^{\tau_n})\,d\langle X\rangle_s$ $\forall n$. Send $n\to\infty \Rightarrow$ QED.

Recall: B a standard d-dim B.M. Then $d\langle B^{(i)}, B^{(j)}\rangle_t = \delta_{ij}\,dt$ [You check].

Eg: Let B be a d-dim B.M. & $W_t^x = x + B_t$, $D \subseteq \mathbb{R}^d$ be a bdd C^2 domain, $\tau = \tau^x =$ exit time of W^x from D. Let $u(x) = E\tau^x = E^x\tau$. Then $-\frac{1}{2}\Delta u = 1$ in D & $u = 0$ on ∂D.

Pf: Suppose $\exists\, v \in C_b^2(D)\cap C(\bar{D})$ is a solution of $-\frac{1}{2}\Delta v = 1$ in D with $v = 0$ on ∂D.

Let $X_t = v(W_{\tau\wedge t}) + \tau\wedge t$. **Claim:** $X \in \mathcal{M}_c$!

Pf: Let $Y_t = (\tau\wedge t, W_{\tau\wedge t})$. Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be defined by $f(x_0, x) = v(x) + x_0$.

Then $X_t = f(Y_t)$. Itô $\Rightarrow$ $dX_t = \sum_0^d \partial_i f(Y_t)\,dY_t^{(i)} + \frac{1}{2}\sum_{i,j=0}^d \partial_{ij}^2 f(Y_t)\,d\langle Y^{(i)}, Y^{(j)}\rangle_t$

$$= 1\,\chi_{\{t\le\tau\}}\,dt + \sum_1^d \partial_i v(W_{\tau\wedge t})\,dW_{\tau\wedge t}^{(i)} + \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j v(W_{\tau\wedge t})\,d\langle W^{(i)}, W^{(j)}\rangle_t^\tau$$

B.V terms cancel.

$$= \chi_{\{t\le\tau\}}\,dt + \text{''} + \frac{1}{2}\Delta v(W_{\tau\wedge t})\,\chi_{\{t\le\tau\}}\,dt = \sum_1^d \partial_i v(W_{\tau\wedge t})\,dW_{\tau\wedge t}^{(i)}.$$

$\Rightarrow X_t - X_0 = \sum_1^d \int_0^t \partial_i v(W_{\tau\wedge t})\,dW_{\tau\wedge t}^{(i)} \in \mathcal{M}_{c,loc}$. Since v is bdd $\Rightarrow E\sup_{t\le T}|X_t| < \infty$

$\Rightarrow X \in \mathcal{M}_c$! $\therefore E^x X_t = E^x X_0$ $\forall t$. $E^x X_0 = E\,v(W_0^x) = v(x)$.

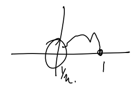
$$\lim_{t\to\infty} E^x X_t = \lim_{t\to\infty} E^x[v(W_{\tau\wedge t}) + \tau\wedge t] \overset{\text{dominated}}{=} E^x v(W_\tau) + E^x\tau = E^x\tau$$

QED.

Recall: X a cts local semi mg. $f \in C^2(\mathbb{R})$. $\forall T$, a.s. $f(X_T) - f(X_0) = \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int^T f''(X_s)\,d\langle X\rangle_s$

Multi D: $f \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$. $df_t(X_t) = \partial_t f_t(X_t)\,dt + \sum_1^d \partial_i f_t(X_t)\,dX_t^{(i)} + \frac{1}{2}\sum_{i,j}\partial_i\partial_j f_t(X_t)\,d\langle X^{(i)}, X^{(j)}\rangle_t$.

Eg: $W = \binom{W^{(1)}}{W^{(2)}}$ a 2D B.M. $M_t = \ln\left|\binom{1}{0} + W_t\right|$. Then $M \in \mathcal{M}_{c,loc}$ & $M \notin M_c$.



Pf: $\tau_n = \inf\{t \mid |e_1 + W_t| < \frac{1}{n}\}$. $f_n = \begin{cases} \ln|x| & |x| > 1/n \\ C^\infty \text{ extension} & |x| \le 1/n \end{cases}$. $M^{\tau_n} = f_n(e_1 + W^{\tau_n})$

$\Rightarrow dM^{\tau_n} = \underbrace{\sum \partial_i f_n(e_1 + W_t^{\tau_n})\,dW^{(i)}_{\tau_n \wedge t}}_{\in \mathcal{M}_{c,loc}.} + \frac{1}{2}\underbrace{\Delta f_n}_{0}\,\chi_{\{t \le \tau_n\}}\,dt$. $\Rightarrow M^{\tau_n} \in \mathcal{M}_{c,loc}$. IOU: $\tau_n \to \infty$ a.s. $\Rightarrow M \in \mathcal{M}_{c,loc}$
$\hookrightarrow EM_t \xrightarrow{t\to\infty} \infty \Rightarrow M \notin M_c$. QED.

Integration by parts: X, Y two cts local s.m. $\int_0^t X_s\,dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s\,dX_s - \langle X, Y\rangle_t$.

Pf: $f(x,y) = xy$. Then $df(X,Y) = \partial_x f(X,Y)\,dX + \partial_y f(X,Y)\,dY + \frac{1}{2}\big(\partial_{xx} f(X,Y)\,d\langle X\rangle + 2\partial_{xy} f(X,Y)\,d\langle X,Y\rangle$
$+ \partial_{yy} f(X,Y)\,d\langle Y\rangle\big)$

$= Y\,dX + X\,dY + d\langle X, Y\rangle$.

$\Rightarrow X_tY_t - X_0Y_0 = f(X_t,Y_t) - f(X_0,Y_0) = \int^t Y_s\,dX_s + X_s\,dY_s + d\langle X,Y\rangle_s$ QED

Stratonovich Integral: Define $\int_0^t Y_s \circ dX_s = \int_0^t Y_s\,dX_s + \frac{1}{2}\langle Y_t, X_t\rangle$. For continuous process Y, one can show $\left(\sum Y_{\frac{t_k + t_{k+1}}{2}}(X_{t_{k+1}} - X_{t_k})\right) \xrightarrow[|\Delta| \to 0]{P} \int_0^T Y \circ dX$.

Remark: While Itô integrals are always (local) martingales, Stratonovich Integrals need not be.

Prop: If $f \in C^3$, then $f(X_t) - f(X_0) = \int_0^t f'(X_s) \circ dX_s$ a.s.

Pf: By Itô, $f(X_t) - f(X_0) = \int_0^t f'(X_s)\,dX_s + \frac{1}{2}f''(X_s)\,d\langle X\rangle_s$
$= \int_0^t f'(X_s) \circ dX_s - \frac{1}{2}\langle f'(X), X\rangle_t + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s$.

By Itô, $df'(X_t) = f''(X_t)\,dX_t + \frac{1}{2}f^{(3)}(X_t)\,d\langle X\rangle_t \Rightarrow \langle f'(X), X\rangle_t = \int_0^t f''(X_s)\,d\langle X\rangle_s$. QED

Def: Given a filtration $\{\mathcal{F}_t\}$, we define the augmentation to be the filtration $\{\sigma(\mathcal{F}_t \cup N)\}$ where $N = \{A \in \mathcal{F}_\infty \mid P(A) = 0\}$. (Rmk the augmented filtration of a strong Markov process is R.C. & hence satisfies the usual conditions)

Thm (Martingale Rep) Let W be a d-dim B.M. & $\{\mathcal{F}_t\}$ be the augmentation of $\{\mathcal{F}_t^W\}$. If $M \in \mathcal{M}^2$, then $\exists X^{(1)} \ldots X^{(d)} \in \mathcal{L}$ $\ni$ $M = \sum I(X^{(i)}, W^{(i)})$. [Hence $M \in \mathcal{M}_c^2$].

Proof: Let $M \in \mathcal{M}_c^2$ be fixed. ($d\langle M\rangle < dt$) Define $\mathcal{M}_*^2 = \{I(X) \mid X \in \mathcal{L}(M)\} \subseteq \mathcal{M}_c^2 \subseteq \mathcal{M}^2$.   ($I(X) = I(X, M)$)

$\forall N \in \mathcal{M}^2$, $\exists N^* \in \mathcal{M}_*^2$ & $Z \in \mathcal{M}^2$ $\ni$ $N = N^* + Z$ & $\forall M_* \in \mathcal{M}_*^2$, $ZM_* \in \mathcal{M}$ (Morally Z orth to everything of $\mathcal{M}_*^2$, i.e. $\langle Z, \varphi_*\rangle = 0$. $\langle , \rangle$ not defined for R.C. proc)

Remark: Above decomp is unique. (Pf: $N = N^* + Z = N^{**} + Z' \Rightarrow \underbrace{N^* - N^{**}}_{\mathcal{M}_*^2} = \underbrace{Z' - Z}_{\perp \text{ to } \mathcal{M}_*^2}$)

Pf: Enough to prove the prop $\forall$ fixed $T > 0$. Let $N \in \mathcal{M}^2$

Itô Isom $\Rightarrow \{I_T(X) \mid X \in \mathcal{L}_T\}$ is a closed subspace of $L^2(\Omega, \mathcal{F}_T)$. $\Rightarrow \exists \varphi \in L^2(\Omega, \mathcal{F}_T)$ & $X \in \mathcal{L}^*$ $\ni$ $N_T = I_T(X) + \varphi$, & $E\,\varphi\, I_T(Y) = 0$ $\forall Y \in \mathcal{L}^*$. Define Z to be an RCLL modification of $\{E_t\,\varphi\}_{t \le T}$. Then certainly $N_t = I_t(X) + Z_t$, & NTS $Z \perp \mathcal{M}_*^2$

i.e. NTS $\forall M^* \in \mathcal{M}_*^2$, $ZM^* \in \mathcal{M}$ (morally $\langle Z, M^*\rangle = 0$). Let $s < t \le T$. Then $E_s(Z_tM_t^*)$

$= E_s\big(Z_tM_s^* + Z_t(M_t^* - M_s^*)\big) = Z_sM_s^* + E_s\big(Z_t(M_t^* - M_s^*)\big)$

Claim: $E\big(Z_t(M_t^* - M_s^*) \mid \mathcal{F}_s\big) = 0$. (Claim $\Rightarrow$ QED) **Pf:** Say $M^* = I(Y, M)$. Enough to show $\forall A \in \mathcal{F}_s$, $E\big(\chi_A Z_t(M_t^* - M_s^*)\big) = 0$. But $E\, Z_t\, \chi_A(M_t^* - M_s^*) = E\, Z_T\, \chi_A \int_s^t Y_r\,dM_r$

$= E\, Z_T \int_0^T \chi_{[s,t]}\chi_A Y_r\,dM_r = 0$ by def of Z. QED.

Claim: Let $Z \in \mathcal{M}^2$. Then $Z\,I(Y, M) \in \mathcal{M}$ $\forall Y \in \mathcal{L}(M)$ $\Longleftrightarrow$ $ZM \in \mathcal{M}$

Remark If $Z \in \mathcal{M}_c^2$, then the claim is trivial: $Z\,I(Y,M) \in \mathcal{M} \Leftrightarrow \langle Z, I(Y,M)\rangle = 0 \Leftrightarrow \int Y\,d\langle Z, M\rangle = 0$

Pf of Claim: Say $ZM \in \mathcal{M}$, & $X \in \mathcal{L}_0$. Let $s, t \in [t_i, t_{i+1})$, $s < t$.

$E_s\big(Z_tX_{t_i}(M_t - M_{t_i})\big) = X_{t_i}E_s(Z_tM_t - Z_tM_{t_i}) = X_{t_i}(Z_sM_s - Z_sM_{t_i}) = Z_sX_{t_i}(M_s - M_{t_i})$

If $s \le t_i \le t$ then $E\big(Z_tX_{t_i}(M_t - M_{t_i}) \mid \mathcal{F}_s\big) = E\big(E(\,(\,)\mid \mathcal{F}_{t_i}) \mid \mathcal{F}_s\big) = 0$

Consequently, $\{Z_t \sum X_{t_i}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})\} \in \mathcal{M}$ $\therefore$ $Z\,I(X) \in \mathcal{M}$ $\forall X \in \mathcal{L}_0$.

Let $(X^{(n)}) \xrightarrow{\mathcal{L}} X$ with $X^{(n)} \in \mathcal{L}_0$ $\forall n$. Then $\forall t$, $(I_t(X^{(n)})) \xrightarrow{L^2} I_t(X)$ & hence $(Z_t I_t(X^{(n)})) \xrightarrow{L^1(\Omega, \mathcal{F}_t)} Z_t I_t(X)$. $\therefore$ $E_s(Z_tI_t(X)) = \lim E_s(Z_t I_t(X^{(n)})) =$

$\lim Z_s I_s(X^{(n)}) = Z_s I_s(X)$ QED.

Goal: Mg-Rep: $\mathcal{F}_t = \mathcal{F}_t^W$ augmented. $M \in \mathcal{M}^2 \Rightarrow M = \sum I(X^{(i)}, W^{(i)})$.

Last time: ① $M \in \mathcal{M}^2$ fixed. $d\langle M\rangle \ll dt$. $\mathcal{M}^2_* = \{I(X) \mid X \in \mathcal{L}(M)\}$. $\forall N \in \mathcal{M}^2$, $N = M_* + Z$, $M_* \in \mathcal{M}^2_*$, $Z \perp \mathcal{M}^2_*$

($Z \perp \mathcal{M}^2_*$ means $\forall Y \in \mathcal{L}(M)$, $Z I(Y,M) \in \mathcal{M}$ (Morally $\langle Z, Y\rangle = 0$, but $\langle , \rangle$ not defined).

② $Z \perp \mathcal{M}^2_* \iff Z \perp M$ (i.e. $Z I(Y,M) \in \mathcal{M}\ \forall Y \in \mathcal{L}(M) \iff ZM \in \mathcal{M}$)

Cor: Let W be a d-dim B.M. Then $\forall N \in \mathcal{M}^2$, $\exists X \in (\mathcal{L})^d$ & $Z \in \mathcal{M}^2$ ∋ $N = Z + \sum I(X^{(i)}, W^{(i)})$

$Z I(Y, W^{(j)}) \in \mathcal{M}$

where $\cancel{\langle Z, I(Y, W^{(j)})\rangle = 0}$ $\forall Y \in \mathcal{L}(W^{(j)})$. [Note $\langle Z, I(Y,W^{(j)})\rangle = 0\ \forall Y \iff ZW^{(j)} \in \mathcal{M}$]

Pf: By induction. Already done for $n=1$. Now pick $N \in \mathcal{M}^2$. Know $\exists X^{(1)} \dots X^{(n-1)}$ & Z' ∋

$N = Z' + \sum_1^{n-1} I(X^{(i)}, W^{(i)})$ ∋ $Z' W^{(i)} \in \mathcal{M}$ $\forall i<n$. Since $Z' \in \mathcal{M}^2$, $\exists X^{(n)} \in \mathcal{L}$ & $Z \in \mathcal{M}^2$ ∋

$Z' = Z + I(X^{(n)}, W^{(n)})$. & $ZW^{(n)} \in \mathcal{M} \Rightarrow \forall i<n$, $ZW^{(i)} = Z'W^{(i)} - I(X^{(n)}, W^{(n)}) W^{(i)} \in \mathcal{M}$

Also $N = Z + \sum_1^n I(X^{(i)}, W^{(i)})$, by def. QED

Thm (Martingale Rep) Let W be a d-dim B.M. & $\{\mathcal{F}_t\}$ be the augmentation of $\{\mathcal{F}_t^W\}$.

If $M \in \mathcal{M}^2$, then $\exists X^{(1)} \dots X^{(d)} \in \mathcal{L}$ ∋ $M = \sum I(X^{(i)}, W^{(i)})$. [Hence $M \in \mathcal{M}^2_c$].

Remark: $X^{(i)}$'s above are unique. (Pf: If $M = \sum I(X^{(i)}, W^{(i)}) = \sum I(Y^{(i)}, W^{(i)})$ then

$\sum I(X^{(i)} - Y^{(i)}, W^{(i)}) = 0 \Rightarrow \forall t\ \sum \int_0^t |X_s^{(i)} - Y_s^{(i)}|^2 ds = 0 \Rightarrow X = Y$)

Pf: Know $M = \sum I(X^{(i)}, W^{(i)}) + Z$, where $ZW^{(i)} \in \mathcal{M}$ $\forall i$. So only NTS $Z = 0$.

Enough to show that for $0 \le t_1 < \dots t_n \le t$, $E Z_t f_1(W_{t_1}) \cdots f_n(W_{t_n}) = 0$ (*) $\forall f_1, \dots f_n \in C_b^2(\mathbb{R}^n)$

Lemma: $\forall f \in C_b^2(\mathbb{R}^d)$, $0 \le t_1 \le t_2$, $\exists \varphi \in C_b^{1,2}((t_1,t_2)\times\mathbb{R}^d)$ ∋ $f(W_{t_2}) = \varphi_{t_1}(W_{t_1}) + \int_{t_1}^{t_2} \sum \partial_i \varphi_s(W_s)\, dW_s^{(i)}$

Pf of (*): Induction. $n=1$. $E Z_t f(W_{t_1}) = E Z_{t_1} f(W_{t_1}) = E Z_{t_1}\left(\varphi_0^{(1)}(W_0) + \sum\int_0^{t_1} \partial_i \varphi_s^{(1)}(W_s)\, dW_s^{(i)}\right)$ (Lemma)

$= E Z_0 \varphi_0^{(1)}(W_0) + E Z_{t_1} \sum \int_0^{t_1} \partial_i \varphi_s^{(1)}(W_s)\, dW_s = 0$.

$n=2$: $E Z_t f_1(W_{t_1}) f_2(W_{t_2}) = E Z_t f_1(W_{t_1}) \left(\varphi_{t_1}^{(2)}(W_{t_1}) + \sum \int_{t_1}^{t_2} \partial_i \varphi_s^{(2)}(W_s)\, dW_s^{(i)}\right)$

$= E Z_t f_1(W_{t_1}) \varphi_{t_1}^{(2)}(W_{t_1}) + E f_1(W_{t_1}) E_{t_1}\left(Z_{t_2} \int_{t_1}^{t_2} \varphi_s^{(2)}(W_s)\, dW_s^{(i)}\right)$

$= 0$ (Induction Hyp) $+ E f(W_{t_1}) Z_{t_1} \int_{t_1}^{t_1} (\,)\, dW_s = 0$ QED (Thm)

Pf of Lemma: ① Suppose $\exists \varphi \in C^{1,2}$ ∋ $\partial_t \varphi + \frac12 \Delta\varphi = 0$ for $t \le t_2$ & $\varphi_{t_2} = f$. (**)

Then $f(W_{t_2}) = \varphi_{t_2}(W_{t_2}) = \varphi_{t_1}(W_{t_1}) + \int_{t_1}^{t_2} \partial_s \varphi_s(W_s)\, ds + \sum \int_{t_1}^{t_2} \partial_i \varphi_s(W_s)\, dW_s^{(i)} + \frac12 \int_{t_1}^{t_2} \Delta\varphi_s(W_s)\, ds$

$= \varphi(W_{t_1}) + \int_{t_1}^{t_2} \sum \partial_i \varphi(W_s)\, dW_s^{(i)}$, which is what we want.

② NTS φ satisfying (**) exists. Let $G_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}$. Explicitly compute

& check $\partial_t G = \frac12 \Delta G$. Now set $u(x,t) = f * G_t(x) = \int f(y) G_t(x-y)\, dy$ $(= E^x f(W_t))$

Since $G \in C^\infty((0,\infty)\times\mathbb{R}^d)$ & $f \in C^2$, $u \in C^{1,2}((0,\infty)\times\mathbb{R}^d)$ & $(\partial_t - \frac12\Delta)u = f * (\partial_t - \frac12\Delta) G_t = 0$.

Also, $(G_t) \xrightarrow{t\to0} \delta_0 \Rightarrow u_0 = f$. So Now $\varphi_t(x) = u_{t_2 - t}(x)$ is the desired function. QED.

Remark: $u(x,t) = E^x f(W_t)$ satisfies $\partial_t u - \frac12\Delta u = 0$ with $u_0 = f$ is a special case of the Feynman Kac form.

Cor: (Itô Rep) $\mathcal{F}_t = \mathcal{F}_t^W$ augmented. $\xi \in L^2(\Omega, \mathcal{F}_T)$. Then $\exists! X^{(i)} \in \mathcal{L}$ ∋ $\xi = E\xi + \sum\int_0^T X^{(i)}\, dW^{(i)}$

Pf: Let Z be a R.C. version of $\{E(\xi|\mathcal{F}_t) - E\xi\}_{t\ge0} \Rightarrow \exists X^{(i)}$ ∋ $\xi - E\xi = Z_T = \sum \int_0^T X^{(i)}\, dW^{(i)}$

Uniqueness: $\sum \int_0^T X_t^{(i)}\, dW_t^{(i)} = \sum \int_0^T Y_t^{(i)}\, dW_t^{(i)} \Rightarrow E\int_0^T |X_t - Y_t|^2 dt = 0 \Rightarrow X = Y$. QED.

Cor: $\xi \in L^2(\Omega, \mathcal{F}_\infty) \Rightarrow \exists! X^{(i)} \in \mathcal{L}$ ∋ $E\int_0^\infty (X_s^{(i)})^2 ds < \infty$ ∋ $\xi = E\xi + \int_0^\infty X_s^{(i)}\, dW_s^{(i)}$.

Pf: Without loss $E\xi = 0$. Let M = R.C. version of $E(\xi|\mathcal{F}_t)$. Know $\exists! X^{(i)} \in \mathcal{L}$ ∋ $M_t = \sum\int_0^t X_s^{(i)}\, dW_s^{(i)}$

Jensen $\Rightarrow E M_t^2 = E(E(\xi|\mathcal{F}_t)^2) \le E\xi^2 \Rightarrow E\int_0^\infty (X_s^{(i)})^2 ds = \sup_t E\int_0^t (X_s^{(i)})^2 ds = \sup_t EM_t^2 \le E\xi^2 < \infty$.

Also, $\sup_t EM_t^2 < \infty \Rightarrow \{M_t\}$ U.I. $\Rightarrow \lim_{t\to\infty} M_t = M_\infty = \xi$. $\Rightarrow \sum\int_0^\infty X_s^{(i)}\, dW_s^{(i)} = \xi$ QED

Def 1: An adapted process $\{X_t, \mathcal{F}_t\}$ is called a Markov process with i.d. μ if

① $X_0 \sim \mu$ & ② For $s<t$, $A \in \mathcal{B}(\mathbb{R}^d)$, $P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s)$ ← Borel func of X_s

Def 2: A d-dimensional Markov Family is an adapted process $\{X_t, \mathcal{F}_t\}$ along with a family of measures $\{P^x\}_{x\in\mathbb{R}^d}$ ∋ ① $\forall F \in \mathcal{F}$, $x \mapsto P^x(F)$ is universally meas, ② $P^x(X_0 = x) = 1$

③ $P^x(X_t \in A | \mathcal{F}_s) = P^x(X_t \in A | X_s)$ $\forall s<t$ & ④ $P^x(X_{t+h} \in A | X_t = y) = P^y(X_h \in A)$ $P^x X_t^{-1}$ a.e. y