Assignment 6: Assigned Wed 11/20. Due Wed 12/04

1. Let $d \in \mathbb{N}, b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ be bounded, Borel measurable and $\sigma : \mathbb{R}^d \times \mathbb{R}^d$ $[0,\infty) \to \mathbb{R}^{d^2}$ be bounded and uniformly Lipschitz. Suppose further there exists $\lambda > 0$ such that for all $t \ge 0$ and $x, y \in \mathbb{R}^d$ we have

$$\sum_{i,j,k} \sigma_t^{(i,k)}(x) \sigma_t^{(j,k)}(x) y^{(i)} y^{(j)} = |\sigma_t(x)^* y|^2 \ge \lambda |y|^2.$$

Then prove weak existence and uniqueness for the SDE

$$dX_t = b_t(X_t) dt + \sigma_t(X_t) dW_t$$

for any given initial distribution μ .

- 2. Let b, σ be uniformly Lipschitz functions on \mathbb{R}^d , and X be the (unique, strong) solution of the SDE $dX_t = b(X_t) dt + \sigma(X_t) dW_t$ with initial data $X_0 = x$.
 - (a) Show that $\lim_{t \to 0^+} \frac{1}{t} E(X_t x) = b(x)$ and $\lim_{t \to 0^+} \frac{1}{t} E(X_t^{(i)} x^{(i)})(X_t^{(j)} x^{(j)}) = 0$ $\sum_{k} \sigma_{ik}(x) \sigma_{ik}(x).$ (b) Show that for all $\varepsilon > 0$, $\lim_{t \to 0^+} \frac{1}{t} P(|X_t - x| > \varepsilon) = 0.$
- 3. Let X be a d-dimensional diffusion satisfying the SDE

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t$$

where b and σ are time independent and Lipshitz. Let $D \subseteq \mathbb{R}^d$ be a domain, and τ be the exit time of X from D. Suppose

$$u \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times (0,\infty)) \cap C(D \times [0,\infty))$$

satisfies

$$\partial_t u - Lu = 0$$
 in D , $u(x,0) = 1$ in D , $u(x,t) = 0$ on $\partial D \times (0,\infty)$,
where $L = \sum_i b_i \partial_i + \frac{1}{2} \sum_{i,j} a_{i,j} \partial_i \partial_j$, and $a_{i,j} = \sum_k \sigma_{i,k} \sigma_{j,k}$. Show that $u(x,t) = P^x(\tau \ge t)$.

- 4. Let $D \subseteq \mathbb{R}^d$ be a domain, and X be a diffusion on D. Let τ_D be the first exit time of X from D, and assume $E^x \tau_D < \infty$. Show that a continuous function u is X-harmonic if and only if the process $\{u(X_t)\}_{t\geq 0}$ is a continuous martingale. [RECALL: A function $u: D \to \mathbb{R}$ is X-harmonic if u is continuous and $u(x) = E^x u(X_\tau)$ for all stopping times $\tau \leq \tau_D$.]
- 5. Fix a bounded C^2 domain D, and a non-degenerate diffusion X. Let A be the generator of X and L the associated differential operator. We say λ is an eigenvalue of -L if there exists a non-zero $u \in C^2(D) \cap C(\overline{D})$ such that $-\Delta u = \lambda u$ in D, and u = 0 on ∂D . We define λ_0 to be the smallest eigenvalue of -L. Show that

$$\lambda_0 \ge \sup\{\rho \mid \sup_{x \in D} E^x \exp(\rho\tau) < \infty\}.$$

[In fact, equality holds above, but that's a little harder to prove.]