Fourier Transform.

① $f \in L^2_{per}[0,1]$; $e_n(x) = e^{2\pi i n x}$. $a_n = \int_0^1 f \bar{e}_n$ & $f = \sum a_n e_n$.

② Rescale: $f \in L^2_{per}[-\frac{L}{2}, \frac{L}{2}]$. $e_n = \frac{e^{2\pi i \frac{nx}{L}}}{\sqrt{L}}$; $a_n = \int_{-L/2}^{L/2} f \bar{e}_n$; $f = \sum a_n e_n$.

Hold $\frac{n}{L} = \xi$ constant. Send $L \to \infty$: $\hat{f}(\xi) = \lim_{L\to\infty} \sqrt{L} a_n = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$

Inversion? $f(x) = \lim_{L\to\infty} \sum a_n e_n(x) = \lim_{L\to\infty} \sum_n \frac{1}{L} \hat{f}(\frac{n}{L}) e^{2\pi i x \frac{n}{L}} \longrightarrow \int_{\mathbb{R}} \hat{f}(\xi) e^{+2\pi i x \xi} d\xi$

3 Miracles: ① $f \in L^2(\mathbb{R}) \Rightarrow \hat{f}$ is defined! ② $\hat{f} \in L^2$ & ③ $f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{+2\pi i x \xi} d\xi$.

Def: $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$. Define $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ [Note: need $f \in L^1$ to define this].

Def: (More generally) μ a finite Borel meas. $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle} d\mu(x)$.

Basic Properties: ① $(f + \alpha g)^\wedge = \hat{f} + \alpha \hat{g}$ (Linearity)

② Translations: $\tau_x f(y) = f(y-x)$. Then $(\tau_x f)^\wedge(\xi) = \int f(y-x) e^{-2\pi i \langle y, \xi \rangle} dy = e^{-2\pi i x \xi} \hat{f}(\xi)$.

③ Dilations: $\delta_\lambda f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. [Note $\|\delta_\lambda f\|_{L^1} = \|f\|_{L^1}$].

$(\delta_\lambda f)^\wedge(\xi) = \int f(\frac{x}{\lambda}) e^{-2\pi i \langle x, \xi \rangle} \frac{dx}{\lambda^d} = \int f(y) e^{-2\pi i \langle y, \lambda \xi \rangle} dx = \hat{f}(\lambda \xi) = \frac{1}{\lambda^d} (\delta_{1/\lambda} \hat{f})(\xi)$

④ Convolutions: $(f * g)^\wedge(\xi) = \int f(y) g(x-y) e^{-2\pi i \langle x, \xi \rangle} dy\, dx \overset{\text{Fubini}}{=} \hat{f}(\xi) \hat{g}(\xi)$.

⑤ $(1+|x|) f \in L^1 \Rightarrow \hat{f}$ is diff & $\partial_j \hat{f}(\xi) = (-2\pi i x_j f(x))^\wedge(\xi)$

Decay of $\hat{f}$ ↔ regularity of f

Pf: $\frac{1}{h}(\hat{f}(\xi + h e_j) - \hat{f}(\xi)) = \frac{1}{h} \int f(x) (e^{-2\pi i \langle x, \xi + h e_j \rangle} - e^{-2\pi i \langle x, \xi \rangle}) dx$

$|f(x) (e^{-2\pi i \langle x, \xi + h e_j \rangle} - e^{-2\pi i \langle x, \xi \rangle}) \frac{1}{h}| \leq |x_j f(x)| \in L^1$. (ind of h) DCT $\Rightarrow$ QED.

⑥ $f \in C_0^1$, $\partial_j f \in L^1 \Rightarrow (\partial_j f)^\wedge(\xi) = +2\pi i \xi_j \hat{f}(\xi)$ [Pf: $(\partial_j f)^\wedge(\xi) = \int \partial_j f(x) e^{-2\pi i \langle x, \xi \rangle} dx$

$= -\int f(\xi) \partial_j (\ ) dx = 2\pi i x_j \hat{f}(\xi)$]

Thm: (Riemann Lebesgue). $f \in L^1 \Rightarrow \hat{f} \in C_0$. [cts & decays at ∞]. & $\|\hat{f}\|_\infty \leq \|f\|_1$

Pf: DCT $\Rightarrow \hat{f}$ is cts. Decay: $(\tau_x f)^\wedge(\xi) = e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)$. Choose $x \ni e^{2\pi i \langle x, \xi \rangle} = -1$.

[Eg. $x = \frac{\xi}{2|\xi|^2}$]. $\Rightarrow (\tau_x f)^\wedge(\xi) = -\hat{f}(\xi) \iff 2\hat{f}(\xi) = (f - \tau_x f)^\wedge(\xi)$.

$\Rightarrow 2|\hat{f}(\xi)| \leq \|f - \tau_x f\|_{L^1} \longrightarrow 0$ QED.

Thm (Inversion) $f, \hat{f} \in L^1$. Then $f(x) = \int \hat{f}(\xi)\, e^{+2\pi i \langle x,\xi\rangle}\, d\xi$. [$\Rightarrow \exists g$ cts $\ni f = g$ a.e.]

Try #1: $\int \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle} d\xi = \iint f(y)\, e^{-2\pi i \langle y,\xi\rangle} e^{+2\pi i\langle x,\xi\rangle} dy\, d\xi \stackrel{?}{=} \iint f(y)\, e^{+2\pi i \langle x-y,\xi\rangle} dy\, d\xi$.

Can't change the order $f(y) e^{2\pi i \langle x-y,\xi\rangle} \notin L^1(dy \times d\xi)$! [Integral DNC after changing the order].

Try #2: Step 1: Let $G(x) = \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2}$ $(x \in \mathbb{R})$. Compute $\hat{G}$.

You check: $(\partial_j f)^\wedge(\xi) = 2\pi i\, \xi_j \hat{f}(\xi)$ & $\partial_j \hat{f}(\xi) = -2\pi i\, (x_j f)^\wedge(\xi)$

Know $G' = -x G \Rightarrow 2\pi i \xi\, \hat{G}(\xi) = -\left(\frac{-1}{2\pi i}\, \partial_\xi \hat{G}\right) \Leftrightarrow \hat{G}' = -4\pi^2 |\xi|\, \hat{G}$ & $\hat{G}(0) = \int G = 1$

$\Rightarrow \hat{G}(\xi) = e^{-2\pi^2|\xi|^2} = e^{-(2\pi\xi)^2/2} = \delta_{1/2\pi} G(\xi) \cdot \frac{1}{\sqrt{2\pi}} \Rightarrow \hat{\hat{G}}(x) = \frac{1}{\sqrt{2\pi}} (\delta_{1/2\pi} G)^\wedge(x) = \frac{1}{\sqrt{2\pi}} \hat{G}\left(\frac{x}{2\pi}\right) = G(x)$.

Cor. For $x \in \mathbb{R}^d$, $G(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$, $\hat{G}(\xi) = e^{-|2\pi\xi|^2/2}$ & $\hat{\hat{G}} = G$. [$\Rightarrow \int \hat{\hat{G}} = 1$]

Lemma 2: Inversion holds for $\forall f \in L^1 \cap C(\mathbb{R}^d) \ni \hat{f} \in L^1$. [Rmk: all $f \in \mathcal{S}$ satisfy this].

Pf: ⓪ If $f, g \in L^1$ then $\int f \hat{g} = \int \hat{f} g$. [Pf: $\int f(x) g(y) e^{-2\pi i \langle y,x\rangle} dy\, dx$ & Fubini].

① Enough to show $f(0) = \int \hat{f}(\xi)$. [$\because f(x) = \tau_{-x} f(0) = \int (\tau_{-x} f)^\wedge(\xi) = \int e^{+2\pi i \langle x,\xi\rangle} \hat{f}(\xi)\, d\xi$]

② Choose $\varphi = G$. Define $\varphi_\varepsilon = \delta_\varepsilon \varphi = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right)$, Know $\{\varphi_\varepsilon\}$ an a.i.

③ Note $\lim_{\varepsilon \to 0} (\varphi_\varepsilon)^\wedge(\xi) = \lim \hat{\varphi}(\varepsilon\xi) = \hat{\varphi}(0) = \int \varphi = 1$ $\forall \xi$. & $\|\hat{\varphi}_\varepsilon\|_\infty \leq 1$.

$\therefore \int \hat{f} = \lim_{\varepsilon \to 0} \int \hat{f} (\varphi_\varepsilon)^\wedge = \lim_{\varepsilon \to 0} \int f (\varphi_\varepsilon)^{\wedge\wedge} = \lim_{\varepsilon \to 0} \int f\, \delta_\varepsilon(\hat{\hat{\varphi}}) = f(0) \int \hat{\hat{\varphi}} = f(0)$.

QED.

Pf of Thm: $f, \hat{f} \in L^1$. $\varphi_\varepsilon * f \xrightarrow{L^1} f$ & Lemma applies to $\varphi_\varepsilon * f$.

$f(x) \underset{\text{a.e., subseq}}{\longleftarrow} \varphi_\varepsilon * f(x) = \int (\varphi_\varepsilon * f)^\wedge(\xi)\, e^{+2\pi i \langle x,\xi\rangle} d\xi = \int (\varphi_\varepsilon)^\wedge(\xi)\, \hat{f}(\xi)\, e^{2\pi i \langle x,\xi\rangle} d\xi \xrightarrow{\text{DCT}} \int \hat{f}(\xi)\, e^{2\pi i \langle x,\xi\rangle} d\xi$.

QED.

Rmk: $f, \hat{f} \in L^1 \Rightarrow \|f - \varphi_\varepsilon * f\|_\infty \underset{\text{Inversion.}}{\leq} \|\hat{f} - (\varphi_\varepsilon * f)^\wedge\|_{L^1} \longrightarrow 0$.

Rmk: $f, \hat{f} \in L^1 \Rightarrow \hat{\hat{f}}(x) = \int \hat{f}(\xi)\, e^{-2\pi i \langle x,\xi\rangle} d\xi = f(-x)$.

Thm (Plancherel) Let $f, g \in \mathcal{S}$ ($\mathbb{C}$ valued). Then $\int f\bar{g} = \int \hat{f}\,\overline{(\hat{g})}$

Pf: ① Let $f, g \in \mathcal{S}$. You check: $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$ & the map $f \longmapsto \hat{f} : \mathcal{S} \to \mathcal{S}$ is a bij.

Compute $\bar{\hat{g}}(\xi) = \left(\int \overline{g(x)}\, e^{+2\pi i\langle x,\xi\rangle}\, dx\right) = (\bar{g})^\wedge(-\xi)$

$\therefore \int \hat{f}\bar{\hat{g}} = \int \hat{f}(\xi)\,\hat{\bar{g}}(-\xi)\,d\xi = \int \hat{\hat{f}}(\xi)\,\bar{g}(-\xi)\,d\xi = \int f(-\xi)\bar{g}(-\xi)\,d\xi.$ QED.

Cor: Let $Ff = \hat{f}$ $\forall f \in \mathcal{S}$. Then F extends to a bij isometry $L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$

Pf: F is linear. $\|Ff\|_2 = \|f\|_2$ $\forall f \in \mathcal{S}$. $\Rightarrow F : L^2 \to L^2$ an isom.

Surj: $Rf(x) = f(-x)$. $\forall f \in \mathcal{S}$, $F^2 f = Rf$. $\Rightarrow \forall f \in L^2$, $F^2 f = Rf \iff f = RF^2 f$. $\Rightarrow$ surj (& inv). QED

Def: Let $s \geq 0$, & define $H^s(\mathbb{R}^d) = \{ f \in L^2 \mid \int |(1+|\xi|^s)\,\hat{f}(\xi)|^2\, d\xi < \infty \}$.

Define $\|f\|_{H^s}^2 = \int (1+|\xi|^s)^2\, |\hat{f}(\xi)|^2\, d\xi$.

Remk: $s = 0 \Rightarrow H^s = L^2$. $s = 1 \Rightarrow f$ + $f \in L^2$ & f has one "weak deriv" in L^2.

$\left[\because\ (\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)\right]$.

Prop: $f \in H^s$, $s \in (0,1] \Rightarrow \|f - \tau_x f\|_2 \leq C|x|^s \|f\|_{H^s}$ $[C = C(s)$, indep of $f]$.

Pf: $\|f - \tau_x f\|_2 = \|\hat{f} - (\tau_x f)^\wedge\|_2 = \|(1 - e^{-2\pi i\langle x,\xi\rangle})\,\hat{f}(\xi)\|_{L^2} \leq C\,\| |x|^s |\xi|^s \hat{f}(\xi)\|_{L^2} \leq C|x|^s \|f\|_{H^s}$.

checked in L^1. Needs Pf in L^2.

QED

Thm: $f \in H^s$, $s > d/2 \Rightarrow f$ is cts! & further $\|f\|_\infty \leq C\|f\|_{H^s}$ for some $C = C(s)$ indep of f. $[H^s \hookrightarrow C]$.

Pf: Say first $f \in \mathcal{S}$. $f(x) = \int \hat{f}(\xi)\, e^{+2\pi i\langle x,\xi\rangle} d\xi = \int \hat{f}(\xi)\,(1+|\xi|^s)\,\dfrac{e^{(\ )}}{(1+|\xi|^s)}\, d\xi$.

$\leq \|f\|_{H^s} \left(\int \dfrac{1}{(1+|\xi|^s)^2}\, d\xi\right)^{1/2} \leq C_s \|f\|_{H^s}$.

finite for $s > d/2$.

② Let $f_n \in \mathcal{S}$, $f_n \xrightarrow{H^s} f$. $\Rightarrow (f_n)$ Cauchy in H^s. $\Rightarrow (f_n)$ Cauchy in $L^\infty \Rightarrow (f_n) \xrightarrow{\text{unif}} f$. QED.

Cor: $f \in H^s$, $s > n + d/2$, $n \in \mathbb{N} \Rightarrow f \in C^n$ & $\|f\|_{C^n} \leq C\|f\|_{H^s}$.

Pf: Induction + $(\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)$. $\Rightarrow \partial_j f \in H^{s-1}$. QED.

Cor: Let $H = \{(x,y) \mid x \in \mathbb{R}^d, y > 0\}$. $f \in L^2(\mathbb{R}^d)$. $u \in C^2(\mathbb{R}^{d+1})$, $u, \partial_i u, \partial_{ij} u \in L^1 \cap L^2$. $\Delta u = 0$ in H

and $\lim_{y \to 0} u(\cdot, y) = f(\cdot)$ [i.e. $\lim_{y\to 0} \int |u(x,y) - f(x)|^2 dx = 0$]. Then $u \in C^\infty(H)$.

Pf: Let $\hat{u}(\xi,y) = \int u(x,y)\, e^{-2\pi i \langle x,\xi\rangle}\, d\xi$. Write $\Delta = \Delta_x + \partial_y^2$

Then $-4\pi^2|\xi|^2 \hat{u} + \partial_y^2 \hat{u} = 0$. Solve: $\hat{u}(\xi,y) = \hat{f}(\xi)\, e^{-2\pi|\xi| y}$ $[\, e^{+2\pi y} \to \infty \text{ as } y \to \infty \,]$

i.e. $u(x,y) = \left(e^{-2\pi|\xi|y}\hat{f}\right)^{\vee}(x,y)$. Note $u(\cdot,y) \xrightarrow{L^2} f$ since $\hat{u}(\cdot,y) \xrightarrow{L^2} \hat{f}$

Also, $\forall y>0$, $u(\cdot,y) \in H^s(\mathbb{R}^d)$ $\forall s>0$. $[\because e^{-|\xi|y}(1+|\xi|)^N \to 0 \;\forall N]$.

$\Rightarrow u$ is inf diff in x. $e^{-2\pi|\xi|y}$ inf diff in $y \Longrightarrow$ QED.