Dynkin Systems.

Motivation: Say $\mu=\nu$ on $\mathcal{G}\subseteq\mathcal{P}(X)$. Must $\mu=\nu$ on $\sigma(\mathcal{G})$?

① Clearly need $\mathcal{G}$ is closed under intersections. [if not  is a counter eg]

② Let $\Sigma=\{A \mid \mu(A)=\nu(A)\}$. What can we show about Σ.

ⓐ Suppose $X\in\Sigma$. $\Rightarrow$ Σ is closed under complements. (provided $\mu(X)<\infty$)

ⓑ Say $A,B\in\Sigma$. $?\Rightarrow$ $A\cup B\in\Sigma$. (Not directly possible – disjoint unions OK)

ⓒ $A\subseteq B\in\Sigma$. $\Rightarrow$ $B-A\in\Sigma$. $\left(B-A=(B^c\cup A)^c\right)$

ⓓ $A_i\in\Sigma$ & $A_i\subseteq A_{i+1}$ $\Rightarrow$ $\bigcup_1^\infty A_i\in\Sigma$.

Dynkin Systems: $\Lambda\subseteq\mathcal{P}(X)$ is a λ-system if ① $X\in\Lambda$, ② $A\subseteq B$ & $A,B\in\Lambda\Rightarrow B-A\in\Lambda$

& ③ $A_n\in\Lambda$, $A_n\subseteq A_{n+1}\Rightarrow\bigcup_1^\infty A_n\in\Lambda$ (aka d-system / Dynkin System).

Def: π-system: $\Pi\subseteq\mathcal{P}(X)$ is a π-system if $A,B\in\Pi\Rightarrow A\cap B\in\Pi$.

Prop: Let Π be a π-system & $\Lambda\supseteq\Pi$ a Λ system. Then $\Lambda\supseteq\sigma(\Pi)$.

Note: Arb int of λ-systems is a λ system. So makes sense to talk about $\lambda(\Pi)$ as the smallest λ system containing Π.

Pf of Prop: Claim: $\lambda(\Pi)=\sigma(\Pi)$. (Claim $\Rightarrow$ Prop)

Obs 1: Any λ system that is also a π-system is a σ-alg. $\}$ Exercise

(Pf: $A,B\in\Lambda$. $A\cup B=\left(A-(A\cap B)\right)\underset{\text{disj}}{\cup}B=\left(B^c-(A-(A\cap B))\right)^c$)

Only NTS $\lambda(\Pi)$ is closed under intersections.

Let $C\in\Pi$, & set $\Lambda'=\{B\in\lambda(\Pi)\mid B\cap C\in\lambda(\Pi)\}$. Claim: Λ' is a λ sys.

Pf: ① $X\in\Lambda'$. ② $A,B\in\Lambda'$ & $A\subseteq B$. Then $(B-A)\cap C=(B\cap C-A\cap C)\in\Lambda'$

③ $A_i\subseteq A_{i+1}\in\Lambda'$. $(\bigcup A_i)\cap C=\bigcup(A_i\cap C)\in\Lambda'$. ✓

$\Rightarrow$ Claim: $\Lambda'\supseteq\Pi$. $\Rightarrow$ $\Lambda'=\lambda(\Pi)$.

Let $C\in\lambda(\Pi)$, & $\Lambda''=\{A\in\lambda(\Pi)\mid C\cap A\in\lambda(\Pi)\}$.

① $\Lambda''\supseteq\Pi$ from above. ② Λ'' a λ-sys (same proof). $\Rightarrow$ $\Lambda''=\lambda(\Pi)$.

$\Rightarrow$ $\lambda(\Pi)$ is closed under intersections. $\Rightarrow$ QED.

Put as HW. ① $\mu=\nu$ on $\mathcal{G}$ & σ-finite $\Rightarrow$ $\mu=\nu$ on $\sigma(\mathcal{G})$.

(Counterexample when μ & ν are not σ-finite).

② A λ-system that is closed under intersections is a σ-alg.