Assignment 12: Assigned Wed 04/23. Due Wed 04/30

1. *(Hopf lemma)* Here's an outline to solve the optional challenge from HW12.
   (a) Given $0 < R_0 < R_1$, let $A(R_0, R_1)$ be the annulus $\{x \in \mathbb{R}^2 \mid R_0 < |x| < R_1\}$. Let $c_0, c_1 \in \mathbb{R}$ with $c_0 < c_1$, and suppose v satisfies the PDE $-\triangle v = 0$ in $A(R_0, R_1)$, with $v = c_0$ on the inner boundary, and $v = c_1$ on the outer boundary. Show that $\partial_r v(R_1, \theta) > 0$. [Hint: Find the solution explicitly.]
   (b) Let $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Suppose u is some function such that $-\triangle u \leqslant 0$ in B_R, and u attains a maximum at some point $x_0 \in \partial B_R$. Suppose further $u(0) < u(x_0)$. Show that $\partial_r u(x_0) > 0$. [Hint: Observe first that for some R_0 small enough, $c_0 = \max_{|x|=R_0} < u(x_0)$. Let $c_1 = u(x_0)$ and use the maximum principle and previous subpart.]
   (c) *(Hopf lemma)* Suppose $D \subseteq \mathbb{R}^2$ is a domain with a smooth boundary. Suppose u is a non-constant function satisfying $-\triangle u \leqslant 0$ in D, and is continuous up to the boundary of D. If u attains it's maximum at a point $x_0 \in \partial D$, show that $\frac{\partial u}{\partial \hat{n}} > 0$ at x_0, where $\hat{n}$ is the outward pointing unit normal vector. [I stated this in 2D for simplicity; with minor modifications to part (a), your proof will work essentially unchanged in 3D as well.]
2. **Sec. 7.1.** 6.
3. **Sec. 7.3.** 2.
4. **Sec. 7.4.** 12, 21, 22.