21-372 PDE: Midterm 2.

Mar 23th, 2012

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 40 points.*
- *You may use without proof any result that has been proved in class or on the homework, provided you **CLEARLY** state the result you are using.*
- *You may assume that all mixed partials are equal, and differentiate under the integral without justification.*
- *Difficulty wise: #1 ≈ #2 ≈ #3 < #4. Good luck!*

[10] 1. Let f be the function defined by $f(x) = 1$ if $x > 0$, $f(0) = 0$, and $f(x) = -1$ if $x < 0$. Let u be a solution of the PDE $\partial_t u - \frac{1}{2}\partial_x^2 u = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = f(x)$. Find a formula for $u(x, t)$, and express your answer in terms of the error function.

[10] 2. Let $X_n(x) = \sin(\frac{n\pi}{L}x)$, and suppose $f(x) = B_1X_1(x) + B_2X_2(x) + B_3X_3(x)$ for some constants B_1, B_2 and B_3. Compute $\int_0^L f(x)^2\,dx$ in terms of B_1, B_2, B_3 and L.

3. Please only do **ONE** of the two following subparts. The second subpart is the higher dimensional analogue of the first, and is worth more points. **You will only get credit for ONE of these two subparts, so please don't do them both.**

[6] (a) Let $L > 0$, and a be some given function defined on $[0, L]$. Must we necessarily have

$$\int_0^L \left[a\partial_x^2 u + (\partial_x a)(\partial_x u)\right] v\,dx = \int_0^L u\left[a\partial_x^2 v + (\partial_x a)\cdot(\partial_x v)\right] dx$$

for all functions u, v such that $u(0) = v(0) = u(L) = v(L) = 0$. Justify your answer. You may assume that the functions a, u, and v are infinitely differentiable.

[10] (b) Let $D \subseteq \mathbb{R}^2$ be a bounded domain, and a be some given function defined on D. Must we necessarily have

$$\int_D \left[a\triangle u + (\nabla a)\cdot(\nabla u)\right] v\,dx\,dy = \int_D u\left[a\triangle v + (\nabla a)\cdot(\nabla v)\right] dx\,dy$$

for all functions u, v which are 0 on the boundary of D. Justify your answer. You may assume that the functions a, u, and v are as infinitely differentiable.

The next question is a little tricky. While the correct solution is very clean and short, arriving at the solution given what you've seen so far isn't too easy.

[10] 4. Let f be a function such that $\int_0^L f(x)\,dx < \infty$. Define as usual

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,dx.$$

Show that

$$\frac{L}{2}(B_1^2 + B_2^2 + B_3^2) \leqslant \int_0^L f(x)^2\,dx.$$

[This is known as Bessel's inequality. You may not use Parseval's identity which I stated, but have not yet proved in class.]