880 Stochastic Calculus: Midterm.

Oct 17^{th}

- This is a closed book test. No calculators or computational aids are allowed.
- You have 80 mins. The exam has a total of 4 questions and 20 points.
- You may use without proof any result that has been proved in class or on the homework, unless you are explicitly instructed otherwise. You must also **CLEARLY** state the result you want to use.

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \ge 0}$, and is always assumed to satisfy the usual conditions.

- 5 1. Let sign⁺(x) = 1 if $x \ge 0$, and 0 if x < 0. Let $Y_t = \int_0^t \operatorname{sign}^+(W_s) dW_s$. Find $P(Y_1 > 2)$. [You may leave your answer as an unsimplified *deterministic* integral, provided you have explicitly computed the integrand.]
- 5 2. Let λ denote the Lebesgue measure on \mathbb{R} . Define the random variable $X: \Omega \to \mathbb{R}$ by

$$X(\omega) \stackrel{\text{def}}{=} \lambda \{ t \ge 0 \mid W_t(\omega) = 0 \}.$$

Compute the distribution of X.

- 5 3. Let $X \stackrel{\text{def}}{=} \{X_t, \mathcal{F}_t \mid 0 \leq t < \infty\}$ be a non-negative, right continuous super-martingale. Show $X_{\infty} \stackrel{\text{def}}{=} \lim_{t \to \infty} X_t$ exists almost surely. Must X_{∞} be integrable? Must $E(X_{\infty} \mid \mathcal{F}_t) \leq X_t$? Prove or find a counter example. [This was a question on HW#1. Please provide a complete solution here, and don't simply say "done on homework".]
- 5 4. Let W be a 3D Brownian motion starting at $0, e_1 = (1, 0, 0) \in \mathbb{R}^3$, and $N : \mathbb{R}^3 \{0\} \to \mathbb{R}$ be defined by $N(x) = \frac{1}{|x|}$. Is the process X_t defined by $N(W_t + e_1)$ a local martingale? Is it a martingale? Prove or disprove. [NOTE: Let $\tau_n = \inf\{t \ge 0 \mid |W_t + e_1| = 1/n\}$. You may assume (without proof) that $\tau_n \to \infty$ almost surely.]