Assignment 6: Assigned Wed 11/28. Due TBA

1. Let $d \in \mathbb{N}$, $b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ be bounded, Borel measurable and $\sigma : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^{d^2}$ be bounded and uniformly Lipschitz. Suppose further there exists $\lambda > 0$ such that for all $t \geqslant 0$ and $x, y \in \mathbb{R}^d$ we have

$$\sum_{i,j,k} \sigma_t^{(i,k)}(x)\sigma_t^{(j,k)}(x)y^{(i)}y^{(j)} = |\sigma_t(x)^* y|^2 \geqslant \lambda |y|^2.$$

Then prove weak existence and uniqueness for the SDE

$$dX_t = b_t(X_t)\, dt + \sigma_t(X_t) dW_t$$

for any given initial distribution μ.

2. Let b, σ be uniformly Lipschitz functions on $\mathbb{R}^d$, and X be the (unique, strong) solution of the SDE $dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t$ with initial data $X_0 = x$.
   (a) Show that $\lim_{t \to 0^+} \frac{1}{t} E(X_t - x) = b(x)$ and $\lim_{t \to 0^+} \frac{1}{t} E(X_t^{(i)} - x^{(i)})(X_t^{(j)} - x^{(j)}) = \sum_k \sigma_{ik}(x)\sigma_{jk}(x)$.
   (b) Show that for all $\varepsilon > 0$, $\lim_{t \to 0^+} \frac{1}{t} P(|X_t - x| > \varepsilon) = 0$.

3. Let X be a d-dimensional diffusion satisfying the SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t) dW_t,$$

where b and σ are time independent and Lipshitz. Let $D \subseteq \mathbb{R}^d$ be a domain, and τ be the exit time of X from D. Suppose

$$u \in C^{2,1}(D \times (0, \infty)) \cap C(\bar{D} \times (0, \infty)) \cap C(D \times [0, \infty))$$

satisfies

$$\partial_t u - Lu = 0 \quad \text{in } D, \qquad u(x, 0) = 1 \quad \text{in } D, \qquad u(x, t) = 0 \quad \text{on } \partial D \times (0, \infty),$$

where $L = \sum_i b_i \partial_i + \frac{1}{2} \sum_{i,j} a_{i,j} \partial_i \partial_j$, and $a_{i,j} = \sum_k \sigma_{i,k}\sigma_{j,k}$. Show that $u(x, t) = P^x(\tau \geqslant t)$.