21-372 PDE: Midterm 1.

Feb 17^{th}, 2012

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 40 points.*
- *You may use without proof any result that has been proved in class or on the homework, provided you **CLEARLY** state the result you are using.*
- *You may assume that all mixed partials are equal, and differentiate under the integral without justification.*
- *Difficulty wise: #1 ⩽ #2 ⩽ #3 < #4 (the last inequality is strict). Good luck!*

1. Let $c \in \mathbb{R}$ be a non-zero constant, and suppose u solves the wave equation

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \quad \text{for } x \in \mathbb{R}, t > 0,$$

with initial data $u(x,0) = \varphi(x)$, and $\partial_t u(x,0) = \psi(x)$.

[2] (a) Write down a formula expressing $u(x,t)$ in terms of φ and ψ. [No proof, or justification is required.]

[8] (b) Suppose φ and ψ are odd functions (i.e. $\varphi(-x) = -\varphi(x)$, and $\psi(-x) = -\psi(x)$). For every $t > 0$, compute $u(0,t)$. [Your answer should be a number, and not involve ψ or φ.]

[5] 2. (a) Find the characteristics of the PDE

$$x\partial_x u + 2y\partial_y u = 0$$

[5] (b) Find the general form of solutions to the PDE above that are defined (and continuous) at all points $(x,y) \in \mathbb{R}^2$.

[10] 3. Suppose u is a solution of the PDE

$$\partial_t u + u\partial_x u = 1 \qquad \text{for } x \in (0,1),\ t > 0,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad \text{for } t > 0.$$

Let

$$a = \int_0^1 u(x,0)\,dx, \quad b = \int_0^1 u(x,0)^2\,dx, \quad \text{and} \quad E(t) = \int_0^1 u(x,t)^2\,dx.$$

Compute $E(t)$ explicitly as a function of t, a and b. [Warning: The method of characteristics does NOT help here. Hint: Compute $\frac{dE}{dt}$, and use it to compute $\frac{d^2E}{dt^2}$.]

[10] 4. Let u be a solution of the heat equation

$$\partial_t u - \partial_x^2 u = 0, \qquad \text{for } x \in (-1,1),\ t \geqslant 0,$$

with initial data $u(x,0) = 1 - x^2$ and Dirichlet boundary conditions

$$u(-1,t) = u(1,t) = 0 \qquad \text{for } t > 0.$$

Show that for every $x \in (-1,1)$, the function $f(t) = u(x,t)$ is non-increasing as a function of t.