Assignment 11: Assigned Wed 04/04. Due Wed 04/11

1. **Sec. 2.10.** 2, 5, 8, 9.
2. (a) In class we only proved the inverse function theorem assuming $Df_a = I$. Prove the theorem if Df_a is any general (invertible) linear transformation. [Don't reinvent the wheel. Reduce it the case we already did.]

   (b) In class we proved that the inverse function g is differentiable. Prove that g is actually C^1.

We've seen the two dimensional version of the implicit function theorem in class. The higher dimensional analogue is a little more messy to write down, but contains essentially the same idea.

3. Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ be open, and suppose $f : U \times V \to \mathbb{R}^n$ is C^1. Let $x_0 \in U$, $y_0 \in V$ and $a = f(x_0, y_0)$. Suppose that the minor obtained by taking all the rows and the last n columns of $Df_{(x_0,y_0)}$ is an invertible matrix. Show that there exists $\varepsilon, \delta > 0$ and a C^1 function $g : B_\delta(x_0) \to V$ such that

$$\{f = a\} \cap B_\varepsilon(x_0, y_0) = \{(x, g(x)) \mid |x - x_0| < \varepsilon\}.$$

   [As we had in class, this shows that $y = g(x)$ is locally the unique solution of the equation $f(x, y) = a$.]

We "un-rigorously" proved a long time ago that the gradient of a function is perpendicular to level sets. With the implicit function theorem, we can make this all rigorous now. The next two problems do this.

4. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a C^1 function, and $S = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ be the graph of f. Let $x_0 \in \mathbb{R}^m$, and $s_0 = (x_0, f(x_0)) \in S$. We define the *tangent space* of S at the point S_0 by

$$TS_{s_0} = \{(x_0 + h, f(x_0) + Df_{x_0}(h)) \mid h \in \mathbb{R}^m\}$$

   (a) If $x_0 = f(x_0) = 0$, show that TS_{s_0} is a subspace of $\mathbb{R}^{m+n}$. What is the dimension of TS_{s_0}? [If x_0 or $f(x_0)$ are non-zero, then TS_{s_0} is itself not a subspace, however it is a translate of a subspace. Namely, if you shift your origin to the point $(x_0, f(x_0))$, then TS_{s_0} becomes a subspace.]

   (b) As an example, let $f(x, y) = x^2 + 2xy$, $(x_0, y_0) = (1, 0)$. Find a subspace $V \subseteq \mathbb{R}^3$, such that $TS_{s_0} = (1, 0, 1) + V$. Also find a basis of V.

5. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1, $s_0 = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$, $f(x_0, y_0) = a$, and the last n columns of Df_{x_0,y_0} form an invertible matrix. Let S be the level set $\{f = a\}$. By the implicit function theorem, S is locally graph $\{(x, g(x)) \mid x \in B_\delta(x_0)\}$ for some C^1 function $g : \mathbb{R}^m \to \mathbb{R}^n$. Since S is locally a graph of a C^1 function, the previous problem defines the tangent space TS_{s_0} at the point s_0. Show that for all $i \in \{1, \dots, n\}$, the vector $\nabla f_i(x_0, y_0)$ is perpendicular to TS_{s_0}. [Hint: Reduce this to showing that for all $h \in \mathbb{R}^m$, you have $[\nabla f_i(x_0, y_0)] \cdot \binom{h}{Dg_{s_0}(h)} = 0$. Some trickery with the chain rule should help you now.]

Assignment 12: Assigned Wed 04/11. Due Wed 04/18

1. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be C^1, and $a \in \mathbb{R}^m$ be a regular value of g. Assume further that the level set $\{g = a\}$ is non-empty. Show that there exists $\varepsilon > 0$ such for all $b \in B(a, \varepsilon)$, the level set $\{g = b\}$ is also non-empty, and b is a regular value of g.
2. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is C^2. We say the critical point $x_0 \in \mathbb{R}^m$ is *non-degenerate* if the Hessian at x_0 is invertible. We say the critical point x_0 is *isolated* if there exists a small neighbourhood of x_0 where f has no other critical points.

   (a) Show that any non-degenerate critical point of f is isolated.

   (b) Give an example of a function with an isolated critical point which is not non-degenerate.
3. Let $f : \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Suppose 0 is a regular value of g, and the level set $\{g = 0\}$ is non-empty and bounded.

   (a) Let $H(x, \lambda) = f(x) + \lambda \cdot g(x)$, for $\lambda \in \mathbb{R}^m$. If all critical points of H are isolated, show that H can have at most finitely many critical points.

   (b) Let $x_1 \dots x_N$ be all the critical points of H above, ordered so that $f(x_1) \leqslant \cdots \leqslant f(x_N)$. Show that the (global) constrained maximum of f given the constraint $g = 0$ is $f(x_N)$, and the (global) constrained minimum of f given $g = 0$ is $f(x_1)$. [This gives an easily checkable criterion to find the global constrained maximum and minimum.]
4. Let $f(x, y) = y$, and $g(x, y) = y - e^{-x^2}$.

   (a) Let $H(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Compute the critical points of H.

   (b) Show that f attains a constrained maximum given the constraint $g = 0$, and compute the (global, constrained) maximum value.

   (c) Show however that f does not attain a (global) constrained minimum value given the constraint $g = 0$. Why this does not contradict question 3?
5. **Sec. 3.7.** 1, 5, 13, 18.