**Assignment 6:** Assigned Wed 02/22. Due Wed 02/29

1. **Sec. 1.6.** 7. [WARNING: The intermediate value theorem does NOT apply to $f'$, as $f'$ need not be continuous.]
2. Suppose two (oddly shaped) pancakes are placed on a table. Show that with one straight cut of a knife you can cut both of them in half simultaneously. [It turns out that if you place three, oddly shaped, burgers in $\mathbb{R}^3$, you can always cut each of them in half with one straight cut of a knife. This however requires some algebraic topology to prove...]
3. If $f$ is differentiable at $a$, then we know that $f(x) \approx f(a) + (x-a)f'(a)$. The function $f(a) + (x-a)f'(a)$ is a "first order" approximation of $f$. The point of this question is to find higher order approximations of $f$, provided the higher order derivatives of $f$ exist.

   Let $a \in \mathbb{R}$ be fixed, and define $P_{n,f}$, the $n^{\text{th}}$ Taylor approximation of $f$ to be the polynomial $P_{n,f}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.
   (a) For $n \geqslant 1$, show that $P'_{n,f}(x) = P_{n-1,f'}(x)$.
   (b) *(Taylor's theorem.)* Suppose $f$ is $n$ times differentiable at $a$. Show that $\lim\limits_{x\to a} \frac{f(x)-P_{n,f}(x)}{(x-a)^n} = 0$. [HINT: L'Hospital's rule]

   The above says that $f(x) - P_{n,f}(x)$ is "of smaller order" than $(x-a)^n$. We'd expect it to be "of order" $(x-a)^{n+1}$. This is indeed the case, under a stronger assumption.
   (c) *(A.k.a Taylor's theorem.)* Suppose $f$ is $n$ times differentiable at $a$, and $f^{(n)}$ is continuous at $a$. Further, suppose there exists $\varepsilon > 0$ such that $f^{(n)}$ is differentiable on $B(a,\varepsilon) - \{a\}$. For all $x \in B(a,\varepsilon) - \{a\}$, show that there exists $\xi$ between $x$ and $a$ such that $f(x) = P_{n,f}(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$. [HINT: Apply the Cauchy mean value theorem repeatedly to $\frac{f(x)-P_{n,f}(x)}{(x-a)^{n+1}}$.]
4. (a) Suppose $f$ is twice differentiable at $a$, and $f$ attains a local maximum at $a$, show that $f'(a) = 0$ and $f''(a) \leqslant 0$.
   (b) Give an example of a function that is twice differentiable at $a$, has $f'(a) = f''(a) = 0$, however does not attain a local maximum at $a$.
   (c) Suppose $f$ is twice differentiable at $a$, $f'(a) = 0$ and $f''(a) < 0$. Show that $f$ attains a local maximum at $a$.
5. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective. Show that $f$ is strictly monotone. [A function is strictly monotone if it is either always strictly increasing, or always strictly decreasing.]
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective, show that $f^{-1}$ is also continuous.
   (c) *(Optional)* Hard challenge: Find an example of $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous and bijective such that $f^{-1}$ is *NOT* continuous.
6. (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable and bijective, however $f^{-1}$ is *not* differentiable.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, bijective and $f'$ is never 0, then show that $f^{-1}$ is also differentiable.

**Assignment 7:** Assigned Wed 02/29. Due Wed 03/07

1. **Sec. 1.7.** 6, 7
2. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is linear. Show that $f$ is differentiable, and $Df_a(h) = f(h)$. [In other words, the derivative of a linear transformation is itself.]
3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that $Df_a = (f'(a))$. That is, show that the linear transformation $Df_a$ is the $1 \times 1$ matrix $(f'(a))$. [This shows that our notion of derivative for functions depending on more than one variable is a generalization of the one variable definition.]
4. Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$. We saw in class that all partial derivatives of $f$ exist, but $f$ is not differentiable at 0.
   (a) Show further that for any $v \in \mathbb{R}^2$, the directional derivative of $f$ in direction $v$ at the point 0 exists. Compute it.
   (b) Show that $\partial_x f$ and $\partial_y f$ are not continuous.
5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that both $\partial_x f$, $\partial_y f$ exist and are continuous in a neighbourhood of $a$. Show that $f$ is differentiable at $a$. [HINT: Note $f(x+s, y+t) - f(x,y) = f(x+s,y+t) - f(x+s,y) + f(x+s,y) - f(x,y)$, and apply the mean value theorem. Generalizing this to functions $f : \mathbb{R}^n \to \mathbb{R}$ is similar.]
6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a$, and attains a local maximum at $a$. Show that $Df_a$ is the 0 linear transformation.
7. *(Divergence, gradient, curl)* This problem defines three vector derivatives are very useful in practice, and asks you to compute a few elementary properties.
   (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a *scalar* function. We define $\nabla f$ (read as 'gradient f') by $\nabla f = \sum_{i=1}^{n} \partial_i f e_i$. Note $\nabla f$ is a vector, with $i^{\text{th}}$ coordinate $\partial_i f$. Also note that $\nabla f$ is the transpose of the Jacobian of $f$.

   If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable, show that $\nabla(fg) = f\nabla g + g \nabla f$.
   (b) Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a *vector* function. We define $\nabla \cdot u$ (read as 'divergence u') by $\nabla \cdot u = \sum_{i=1}^{n} \partial_i u_i$. Note $\nabla \cdot u$ is a scalar, and equals the trace of the Jacobian of $u$.

   If $f : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}^n$ are differentiable, show that $\nabla \cdot (fu) = (\nabla f) \cdot u + f(\nabla \cdot u)$.
   (c) Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a *vector* function. We define $\nabla \times u$ (read as 'curl u') by $\nabla \times u = \left(\begin{smallmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{smallmatrix}\right)$. Note that $\nabla \times u$ is a 3-dimensional vector.

   If $f : \mathbb{R}^3 \to \mathbb{R}$ and $u : \mathbb{R}^3 \to \mathbb{R}^3$ are differentiable, show that $\nabla \times (fu) = f(\nabla \times u) + (\nabla f) \times u$.
   (d) Suppose $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ are differentiable. Show that $\nabla \cdot (u \times v) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$.