Assignment 4: Assigned Wed 02/08. Due Wed 02/15

1. **Sec. 1.6.** 1.
2. (a) Say $C \subseteq \mathbb{R}$ is closed, non-empty and bounded above. Show $\sup(C) \in C$.

   (b) Say $U \subseteq \mathbb{R}$ is open, non-empty and bounded above. Show $\sup(U) \notin U$.

   (c) If $S \subseteq \mathbb{R}$ is *both* open and closed, show that $S = \mathbb{R}$ or $S = \emptyset$. [Hint: For contradiction, suppose S is open, closed, $S \neq \emptyset$ and $S \neq \mathbb{R}$. Let $\alpha \in \mathbb{R} - S$, and $T = S \cap (-\infty, \alpha)$. Show first that T is both open and closed (this is very short). Next show that $T = \emptyset$ (Hint: this is part (c) of a question.) Now concluding should be quick.]
3. *(Intermediate value theorem)* Let $a < b \in \mathbb{R}$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. If $f(a) < 0$ and $f(b) > 0$, show that there exists $c \in (a, b)$ such that $f(c) = 0$. [Hint: Suppose not. Let $g(x) = f(a)$ if $x < a$, $g(x) = f(b)$ if $x > b$ and $g(x) = f(x)$ otherwise. Then show that the set $\{g < 0\} \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid g(x) < 0\}$ is both open and closed.]
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Show that it has a root in $\mathbb{R}$.
5. Show that a bounded *increasing* sequence of real numbers is convergent. [Recall a sequence (a_n) is called a bounded if $\exists R \mid \forall n \in \mathbb{N},\ |a_n| \leqslant R$. A sequence is increasing if $a_{n+1} \geqslant a_n$ (or equivalently, whenever $m \leqslant n$, we have $a_m \leqslant a_n$). To show that a sequence is convergent, your first task is to identify the limit. If (a_n) is bounded and increasing, a natural guess for the limit is $\sup\{a_n \mid n \in \mathbb{N}\}$! This works, but *only* if your sequence is a *bounded*, *increasing* sequence of *real* numbers.]
6. Let (a_n) be a bounded sequence in $\mathbb{R}$. Define $b_n = \inf\{a_k \mid k \geqslant n\}$, and $c_n = \sup\{a_k \mid k \geqslant n\}$.

   (a) Show that the sequences (b_n) and (c_n) are both convergent. [Hint: I claim (b_n) is bounded and increasing!]

   The limits of (b_n) and (c_n) are usually called the 'lim-inf' and 'lim-sup' of the sequence (a_n). Namely, we define $\liminf a_n = \lim b_n$, and $\limsup a_n = \lim c_n$. Sometimes $\liminf$ and $\limsup$ are denoted by $\underline{\lim}$ and $\overline{\lim}$ respectively. While bounded sequences need not have limits, they always have a lim-sup and a lim-inf. Further, these give a useful criterion for convergence.

   (b) Show that a bounded sequence of real numbers (a_n) is convergent if and only if $\limsup a_n = \liminf a_n$. In this case, show that $\lim a_n = \limsup a_n = \liminf a_n$.

   (c) Show that any Cauchy sequence in $\mathbb{R}$ is convergent. [Hint: This is part (c) of a question. Note that to get convergence of Cauchy sequences in $\mathbb{R}^d$, you only have to observe that each coordinate of the sequence will be a Cauchy sequence (and hence a convergent sequence) in $\mathbb{R}$.]

Assignment 5: Assigned Wed 02/15. Due Wed 02/22

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial. Show that $|f|$ attains a minimum in $\mathbb{R}$.
2. (a) Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, and $f(x) = 0$ if $x = 0$. Show that f is differentiable, however f' is *not* continuous (let alone differentiable).

   (b) Let $n \geqslant 2$. Find a function that is n times differentiable, however the n^{th} derivative of f is not continuous?
3. Suppose f and g are differentiable at a, and $g(a) \neq 0$. Show that f/g is differentiable at a, and $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g(a)^2$.
4. (a) Suppose f and g are n times differentiable at a. Show that fg is also n times differentiable at a, and find a formula for $(fg)^{(n)}(a)$ in terms of derivatives of f and g.

   (b) Let $n \in \mathbb{N}$. Suppose f is n times differentiable at $g(a)$ and g is n times differentiable at a. Show that $f \circ g$ is n times differentiable at a. [Don't try finding a formula for $(f \circ g)^{(n)}$. You won't be able to.]