Homework Assignment 2
Assigned Thu 09/16. Due Wed 10/06.

1. Say $X_t$ is a process with independent increments. Show that if $s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s^X$. 
   [Recall $\mathcal{F}_t^X = \sigma(\cup_{s \leq t} \sigma(X_s))$. Also, $X$ has independent increments means that for any finite sequence $0 \leq t_0 < t_1 \cdots < t_n$, the family of random variables $\{X_{t_j} - X_{t_{j-1}} \mid 1 \leq j \leq n\}$ is independent.]

2. Let $B = \{(B^{(1)}, \ldots, B^{(d)}), \mathcal{F}_t\}$ be a $d$ dimensional Brownian motion starting at $0$. Show that each coordinate $B^{(i)} \in \mathcal{M}^2_c$, and that $\langle B^{(i)}, B^{(j)} \rangle_t = \delta_{ij} t$.

3. Let $B$ be a standard 1D Brownian motion. Show that for $\alpha > \frac{1}{2}$, $\lim_{t \to \infty} \frac{|B_t|}{t^\alpha} = 0$ almost surely. [Don’t use the law of iterated logarithm, as the proof relies on this problem.]

4. Let $B$ be a standard 1D Brownian motion. Show that each of the following are also standard 1D Brownian motions:
   (a) $\{-B_t\}_{t \geq 0}$
   (b) $\left\{ \frac{1}{\sqrt{\lambda}} B_{\lambda t} \right\}_{t \geq 0}$, for any $\lambda > 0$.
   (c) $W_t = t B_{\frac{1}{t}}$ for $t > 0$, and $W_0 = 0$.
   (d) $\{B_{s+t} - B_s\}_{t \geq 0}$ for any fixed $s \geq 0$.

5. Let $B$ be a standard 1D Brownian motion. Show that for any $\alpha > \frac{1}{2}$, $t \geq 0$,

   $$\lim_{h \to 0^+} \frac{|B_{t+h} - B_t|}{h^\alpha} = \infty,$$

   almost surely. [Note that from the previous two problems, you immediately have for $\alpha < \frac{1}{2}$, $\lim_{h \to 0^+} \frac{|B_{t+h} - B_t|}{h^\alpha} = 0$ almost surely.]