Math 341 Syllabus and Lecture schedule.
Gautam Iyer, Fall 2010

L1, Mon 01/10. • Introduction & motivation
  • Fields
    - Definition.
    - Examples. \( \mathbb{R}, \mathbb{Q}, \{0, 1\}, \text{etc.} \) (Also non-examples like \( \mathbb{Z} \)).
    - Multiplication by 0.
L2, Wed 01/12. • Uniqueness of inverses. Inverse of inverses.
L3, Fri 01/14. • Vector spaces
  - ‘Head-to-toe’ addition in \( \mathbb{R}^2 \).
  - Axiomatic definition.
  - Examples: 0, \( \mathbb{R}^n \), \( F^n \), function spaces.
L4, Wed 01/19. • Subspaces, examples.
  - \( U \subseteq V \) a subspace iff \( U \neq \emptyset \) and \( \forall u, v \in U, \alpha \in F, u + \alpha v \in U \).
L5, Fri 01/21. • Span, Linear Independence
  * Define \( \text{span}\{v_1, \ldots, v_n\} \), and show it is a subspace.
  * Define linear independence, linear dependence, and Basis
L6, Mon 01/24. • Example of L.I.
  * Let \( S \) be L.I. Then \( S \cup \{u\} \) is L.I. iff \( u \notin \text{span}(S) \).
L7, Wed 01/26. • Basis and Dimension
  * If \( V \) is spanned by \( n \) vectors, then any \( n+1 \) vectors in \( V \) are linearly dependent.
  * Linearly independent lists can’t be longer than spanning lists.
L8, Fri 01/28. • Any two (finite) basis have the same cardinality.
  * Any L.I. subset of a finitely generated vector space can be extended to a basis.
  * Any finitely generated vector space has a basis.
  * Define dimension of a vector space.
L9, Mon 01/31. • Any subspace of a finitely generated vector space is also finitely generated; and hence has dimension not larger than that of the whole space.
  * Any \( n \) L.I. vectors in a space of dimension \( n \) are also spanning.
  * Any \( n \) spanning vectors in a space of dimension \( n \) are also L.I.
L10, Wed 02/02. • Any vector can be uniquely expressed as a linear combination of basis vectors.
  * Error correcting codes.
    - \( F = \{0, 1\} \), choose \( a, b, c \in F \).
    - Pick \( \{v_1, v_2, v_3\} \in F^n \), and transmit the message \( u = av_1 + bv_2 + cv_3 \).
    - Let \( C = \text{span}\{v_1, v_2, v_3\} \). \( e_i \notin C \) for all \( i \), then one error can be detected.
    - If \( e_i + e_j \notin C \) for \( i \neq j \), then one error can be corrected.
    - E.g. \( n = 6, v_1 = e_1 + e_2 + e_3, v_2 = e_1 + e_4 + e_5, v_3 = e_1 + e_3 + e_4 + e_6 \). Allows you to transmit a 3 bit message using 6 bits, such that an error of at most 1 bit can be corrected!
L11, Fri 02/04. • Fields
  - \( \mathbb{R}, \mathbb{Q}, \{0, 1\}, \text{etc.} \) (Also non-examples like \( \mathbb{Z} \)).
L12, Mon 02/07. • Linear transformations
  - Definition, and some examples.
  - Kernel, image. Proof that \( \ker(T) \) and \( \text{im}(T) \) are subspaces.
  - \( \mathcal{L}(U, V) \), and proof that it is a vector space.
  * Closure of \( \mathcal{L}(U, U) \) under composition.
  * Associativity, non-commutativity, Identity.
  * \( T \in \mathcal{L}(U, V) \) bijective \( \iff \dim(U) = \dim(V) \)
    \( T \in \mathcal{L}(U, V) \) injective \( \iff \ker(T) = \{0\} \)
    \( T \in \mathcal{L}(U, V) \) surjective \( \iff \text{im}(T) = \mathbb{R}^m \)
  * Rank Nullity: \( \dim(\ker(T)) + \dim(\text{im}(T)) = \dim(\text{domain}(T)) \)
    * If \( T \in \mathcal{L}(U, V) \), and \( \dim(U) < \dim(V) \) then \( T \) is not surjective.
    * If \( T \in \mathcal{L}(U, V) \), and \( \dim(U) > \dim(V) \) then \( T \) is not injective.
    * Any system of linear homogeneous equations with more variables than equations has a non-zero solution.
    * Isomorphisms.
  - Matrices
    * Linear transformations can be uniquely determined from values on basis vectors.
    * Define matrix representations of vectors and linear transformations.
    * Compute \( \mathcal{M}_c(Tu) \) in terms of \( \mathcal{M}_{B,C}(u) \) and \( \mathcal{M}_B(u) \).
    * Define \( \text{Mat}(n, m, F) \)
    * Motivate the definition of matrix multiplication by composition of linear transformations.
L13, Fri 02/11. • Fields
L14, Mon 02/14. • Linear equations
  - Elementary row operations don’t change solutions of \( Ax = b \).
L20, Mon 02/28. • Polynomials
- Define $P_F(x)$ to be the ring of formal polynomials.
- Degree, addition, multiplication.
- For $p, q \in P_F(x)$, with $q \neq 0$, $\exists s, r \in P_F(x)$ such that $p = qs + r$ and $\deg(r) < \deg(q)$.

L21, Wed 03/02.
- The evaluation map $E_\alpha$, roots, divisibility.
- If $\alpha$ is a root of $f$ then $(x - \alpha)|f$
- Fundamental theorem of Algebra
- If $f \in P_C(x)$ is non-constant, then $f$ factors completely as a product of linear polynomials.

L22, Mon 03/14. • Eigenvalues
- Definition
- Eigenvalues of diagonal matrices.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- If $\dim(V) < \infty$, then $\lambda$ an eigenvalue of $T \iff T - \lambda I$ is not invertible.

L23, Wed 03/16.
- If $F = \mathbb{C}$, $\dim(V) < \infty$, then every linear transformation from $V$ to $V$ has an eigenvalue.
- Explicitly computing eigenvalues of $2 \times 2$ matrices.

L24, Fri 03/18.
- Tournament ranking / page rank.

L25, Mon 03/21.
- Diagonalizable transformations, matrices.
- Computing powers of diagonalizable matrices.
- Computing Fibonacci numbers as an application.

L26, Fri 03/03.
- Basis change: If $B, C$ are two basis of $V$ then
  - For $x \in V$, $\mathcal{M}_C(x) = \mathcal{M}_C(B)\mathcal{M}_B(x)$.
  - $\mathcal{M}_C(T) = P^{-1}\mathcal{M}_B(T)P$, where $P = \mathcal{M}_B(C)$.

L27, Mon 03/28.
- Upper triangular matrices.
- If $F = \mathbb{C}$, and $\dim(V) < \infty$, then any linear transformation has a basis under which it is upper triangular.

L28, Wed 03/30.
- Invertibility, and eigenvalues of upper triangular matrices.

L29, Fri 04/01. • Inner product spaces
- $(x, y) = \sum x_i y_i = y^t x$ as a generalization of $\|x\|\|y\| \cos \alpha$.
- Using the inner-product to define length: $\|x\| = \sqrt{x, x}$.

L30, Mon 04/04.
- Complex inner products.
- Orthogonality, Pythagoras theorem.
- Cauchy-Schwartz inequality
- Triangle inequality.

L31, Wed 04/06.
- Orthogonal complements.
- Linear independence.
- Coordinates with respect to an orthonormal basis.
- Gram-Schmidt orthogonalization.
- Orthogonal projections.
- Existence, and uniqueness.
- Length minimizing property: If $P$ is the orthogonal projection onto $U$, then $\|v - Pv\| \leq \|v - u\|$ for all $v \in V, u \in U$.

L32, Fri 04/08.
- Introduction to Least squares.
- Minimize error when solving $Ax = b$
- The least squares solution $x_*$ is defined to be the solution of $Ax_* = Pb$, where $P$ is the orthogonal projection onto $\im(A)$.
- The least squares solution can be found by solving the normal equation $A^*Ax_* = A^*b$ (proof next week).

L33, Mon 04/11.
- Orthogonal compliments.
- If $P$ is the orthogonal projection onto $U$, then $U^\perp = \ker(P)$.
- $\dim(U) + \dim(U^\perp) = \dim(V)$.
- For all $v \in V$, there exist unique $u_1 \in U$ and $u_2 \in U^\perp$ such that $v = u_1 + u_2$.

L34, Wed 04/13.
- Duals
- If $\dim(V) < \infty$, and $\varphi \in V^*$ then there exists a unique $v \in V$ such that $\varphi(u) = \langle u, v \rangle$ for all $u \in V$.
- Proof 1: $T : V \to V^*$ by $T(v)(u) = \langle u, v \rangle$, and check that $T$ is (conjugate) linear and injective. This would imply $T$ is surjective, finishing the proof.
- Proof 2: Let $U = \ker(\varphi)$. If $U = V$, then choose $v = 0$. If not, pick any $v_1 \in V$ non-zero, and choose $v = \frac{\varphi(v_1)}{\|v_1\|} v_1$.

L35, Mon 04/18.
- Adjoints
- If $V = \mathbb{R}^n$, with the standard inner product, $A \in \text{Mat}(n, n, \mathbb{R})$, and $Tx = Ax$, then $T^*x = A^t x$.
- If $V = \mathbb{C}^n$, with the standard inner product, $A \in \text{Mat}(n, n, \mathbb{C})$, and $Tx = Ax$, then $T^*x = \overline{A^t} x$.
- If $\dim(V) < \infty$, then $T^*$ exists.
- Uniqueness of the adjoint.

L36, Wed 04/20.
- Self-adjoint operators
- $T = T^* \iff$ all eigenvalues are real.
* $T = T^*$ $\implies$ eigenvectors corresponding to distinct eigenvalues are orthogonal.

- Orthogonal transformations

$L38$, Mon 04/25.

- Normal operators

* $T$ normal $\implies T - \lambda I$ is also normal.
* $T$ normal $\implies \|Tv\| = \|T^*v\|
* $T$ normal and $Tv = \lambda v$ $\implies T^*v = \lambda v$.
* Spectral theorem: $T$ normal, dim$(V) < \infty$, $F = \mathbb{C}$, then there exists an orthonormal basis of $V$ consisting of eigenvectors of $T$. (Proof: Choose an eigenvector $v_1$, put $U = \text{span}\{v_1\}^\perp$, show $T, T^* \in \mathcal{L}(U, U)$, and use induction.)