In this exam, we always assume $V$ is a vector space over a field $F$.

1. Let $F = \mathbb{R}$, and $V$ be the vector space consisting of all real valued functions with domain $[0, 1]$. For any $f \in V$, define $Tf = f(1) - f(0) + 1$. Does $T \in \mathcal{L}(V, \mathbb{R})$ (i.e. is $T$ a linear transformation)? Prove / disprove.

2. Let $M = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$. Find a basis of $\ker(M)$. [Recall $\ker(M) = \{x \in \mathbb{R}^3 \mid Mx = 0\}$]

3. Does there exist $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ such that $\text{im}(T) = \ker(T)$? If yes, find an example. If no, prove it. [If you say ‘Yes’ and produce an example, then you should also prove that your example has the desired properties.]

4. Suppose $T \in \mathcal{L}(V, V)$ is an injective linear transformation such that $T^2 = 4T$. Find all eigenvalues of $T$. (You should prove whatever answer you get.)

5. Let $U \subseteq F^n$ be a subspace with $\dim(U) = m < n$. Show that there exists an $(n - m) \times n$ dimensional matrix $M$ such that $U = \ker(M)$. 