21341 Linear Algebra: Midterm 1.

Wed 02/09

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 minutes. The exam has a total of 5 questions and 50 points.*
- *You may use without proof any result that has been proved in class or on the homework, provided you **CLEARLY** state the result you are using.*
- *Difficulty wise: #1 $\leqslant$ #2 $\approx$ #3 $<$ #4 $<$ #5. The first inequality is not strict. The last two are. (The last inequality should be 'stricter' than the others.)*

In this exam, we always assume V is a vector space over a field F.

[10] 1. Let $F = \mathbb{R}$, $V = \mathbb{R}^3$. Is $\{\left(\begin{smallmatrix}1\\0\\14\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\10\\21\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\17\\21\end{smallmatrix}\right), \left(\begin{smallmatrix}4\\25\\31\end{smallmatrix}\right)\}$ linearly independent? Justify.

[10] 2. Let $n \geqslant 2$, $V = F^n$. Let $u = \begin{pmatrix}u_1\\ \vdots \\ u_n\end{pmatrix} \in V$, $v = \begin{pmatrix}v_1\\ \vdots \\ v_n\end{pmatrix}$. Suppose $u_1 \neq 0$, and for every $i \in \{1, \dots, n\} \in V$, $u_1 v_i - u_i v_1 = 0$. Show that $\{u, v\}$ is linearly dependent.

[10] 3. Let $F = \mathbb{C}$, $V = \mathbb{C}^3$. Let $u = \begin{pmatrix}1+2i\\ 1-i \\ 3+i\end{pmatrix}$, and $v = \begin{pmatrix}-3+4i\\ 3+i \\ 1+7i\end{pmatrix}$. Is the set $\{u, v\}$ linearly independent? Justify.

[10] 4. Let $n \geqslant 2$, and suppose $\alpha_1, \dots, \alpha_n \in F$ are not all 0. Let U be the subspace defined by

$$U = \left\{ \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \in F^n \;\middle|\; \alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \right\}$$

Show that $\dim(U) = n - 1$. [For simplicity, you may assume $\alpha_n \neq 0$. The hint is to find a basis of U consisting of vectors, each having only two, strategically chosen, non-zero coordinates.]

[10] 5. Let $n \geqslant 2$, and $U \subseteq F^n$ be a subspace of dimension $n-1$. Show that there exist $\alpha_1, \dots, \alpha_n \in F$, which are not all 0, such that

$$U = \left\{ \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \in F^n \;\middle|\; \alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \right\}$$