Homework Assignment 7

Assigned Fri 02/25. Due Fri 03/18.

1. (a) Compute the inverse of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \text{Mat}(3, 3, \mathbb{R})$.

   (b) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Find a necessary and sufficient condition for A to be invertible. Find an explicit formula for A^{-1}.

2. Let A be a $m \times n$ matrix. We define the transpose of the matrix A^{t} (sometimes denoted by A^*) to be the matrix obtained by interchanging the rows and columns of the matrix. For example, $\left(\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right)^{\text{t}} = \left(\begin{smallmatrix} a & d \\ b & e \\ c & f \end{smallmatrix}\right)$. If A is an $m \times n$ matrix, B is an $n \times n'$ matrix, show that $(AB)^{\text{t}} = B^{\text{t}} A^{\text{t}}$.

3. Let $T \in \mathcal{L}(U, V)$, $B = \{u_1, \ldots, u_m\}$ an basis of U, and $C = \{v_1, \ldots, v_n\}$ be an basis of V, and let $M = \mathcal{M}_{B,C}(T)$.

   (a) We define the *column rank* of the matrix M to be the maximum number of linearly independent columns of M. That is, treat the columns of M as m vectors in F^n. Then the cardinality of the largest linearly independent subset of these m vectors is defined to be the column rank of M.
   Show that the column rank of M equals $\dim(\text{im}(T))$.

   (b) Similarly, we define the *row rank* of the matrix M to be the maximum number of linearly independent rows of M. Show that the row rank of M equals $m - \dim(\ker(T))$. Show also that $\dim(\ker(T))$ equals the number of free variables in the row reduced echelon form of M.

   (c) Show that the row rank and column rank of M are equal.

4. Let $\{v_1, \ldots, v_n\} \subseteq F^m$, and suppose you want to find a basis for $V = \text{span}\{v_1, \ldots, v_n\}$. We know any spanning set contains a basis. The following algorithm will tell you how to choose a subset $\{v_1, \ldots, v_n\}$ which is a basis of V. First form a matrix with $v_1, \ldots, v_n$ as *columns*. Now put this matrix in *row* reduced echelon form. Now, for each i, if the i^{th} *column* in the reduced matrix is a pivot (i.e. contains a leading 1), pick the vector v_i to be part of your basis. If the column is not a pivot, don't pick the vector. We claim that this algorithm will give you a method to reduce $\{v_1, \ldots, v_n\}$ to a basis of V.

   (a) Using the above algorithm, reduce the following sets to a linearly independent set with the same span.
   (a) $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}$ (b) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -8 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ -5 \\ -1 \end{pmatrix} \right\}$

   (c) Prove that the above algorithm works.

   (d) Explain how you would adapt this algorithm if F^m above was replaced with an abstract (finite dimensional) vector space U.

5. (a) Let $p, q, d \in P_F(x)$. We say d is a common factor of p and q if d is non-constant, $d|p$ and $d|q$. We say p and q have no common factor if whenever $d|p$ and $d|q$, d must be a constant polynomial. If $p, q \in P_F(x)$ have no common factor, show that there exist $a, b \in P_F(x)$ such that $ap + bq = 1$.

   (b) We say a non-constant polynomial $p \in P_F(x)$ is *prime* if whenever $p|fg$, we must have $p|f$ or $p|g$ for every $f, g \in P_F(x)$. We say a non-constant polynomial $q \in P_F(x)$ is *irreducible* if for every $f, g \in P_F(x)$ whenever $q = fg$, either f or g is a constant polynomial. Show that $p \in P_F(x)$ is prime if and only if it is irreducible.

   (c) Find all prime polynomials in $P_{\mathbb{C}}(x)$.

   (d) Find all prime polynomials in $P_{\mathbb{R}}(x)$.