Homework Assignment 6

Assigned Fri 02/18. Due Fri 02/25.

1. Suppose $T \in L(U,V)$, and $\dim(U) = \dim(V) < \infty$. Show that T is injective if and only if T is surjective.

2. (a) Suppose U is a *finite dimensional* vector space over F, and $S, T \in \mathcal{L}(U,U)$ are such that $ST = I$. Show that $TS = I$.

   (b) Show that the previous subpart is *false* if U is not finite dimensional. Namely find an infinite dimensional vector space, and two linear transformations $S, T \in \mathcal{L}(U,U)$ such that $ST = I$ but $TS \neq I$.

3. Let $S, T \in \mathcal{L}(U,U)$. Show that ST is invertible if and only if both S and T are invertible. In this case, express $(ST)^{-1}$ in terms of S^{-1} and T^{-1}.

4. (a) Let U be a finite dimensional vector space over F. Let $B = \{u_1, \dots, u_m\}$ be a basis of U. Let $T : U \to F^m$ be defined by $T(u) = \mathcal{M}_B(u)$ (i.e. $T(u)$ is the matrix representation of u with respect to the basis B). Show that T is an isomorphism.

   (b) Suppose U and V are two finite dimensional vector spaces over F. Show that U and V are isomorphic if and only if $\dim(U) = \dim(V)$. [This result is false if $\dim(U) = \dim(V) = \infty$.]

5. Let U, V be two vector spaces, $B = \{u_1, \dots, u_m\}$ be a basis of U, $V = \{v_1, \dots, v_n\}$ be a basis if V.

   (a) If $S, T \in \mathcal{L}(U,V)$, show that $\mathcal{M}_{B,C}(S+T) = \mathcal{M}_{B,C}(S) + \mathcal{M}_{B,C}(T)$.

   (b) Let $\varphi : \mathcal{L}(U,V) \to \operatorname{Mat}(n,m,F)$ be defined by $\varphi(T) = \mathcal{M}_{B,C}(T)$. Show that φ is an isomorphism. Conclude $\dim \mathcal{L}(U,V) = mn$.

   (c) If W is a vector space, $D = \{w_1, \dots, w_N\}$ a basis of W, $S \in \mathcal{L}(U,V)$, $T \in \mathcal{L}(V,W)$, then show that $\mathcal{M}_{C,D}(T)\mathcal{M}_{B,C}(S) = \mathcal{M}_{B,D}(TS)$.