Homework Assignment 5

Assigned Fri 02/11. Due Fri 02/18.

1. We've defined what it means for a finite (non-empty) set S to be linearly independent. For other sets, we define $S \subseteq V$ to be *linearly independent* if *every* finite non-empty subset of S is linearly independent. Finally, we say S is linearly dependent if it is *not* linearly independent.
   (a) If S is linearly dependent, show that there exists $n \in \mathbb{N}$, $v_1, \dots, v_n \in S$, $\alpha_1, \dots, \alpha_n \in F$ not all 0, such that $\sum_{i=1}^n \alpha_i v_i = 0$.
   (b) With the above definition, is the empty set linearly independent?
   (c) Does the vector space $\{0\}$ have a basis? If yes, what is it?

Suppose one only wants to *detect* errors while transmitting a message. The usual way to do this is to transmit a block of your message (say 7 bits), and then transmit one 'parity bit'. This parity bit is chosen so that the total number of 1's in your message block (including the parity bit) is even. In our setup, this amounts to the following.

2. Let $F = \{0, 1\}$, $n \geqslant 2$, $V = F^n$ and define $C = \operatorname{span}\{e_1 + e_n, e_2 + e_n, \dots, e_{n-1} + e_n\}$. Show that $\dim(C) = n - 1$, and for all $i \in \{1, \dots, n\}$, $e_i \notin C$.

3. *(Optional. Try it, but don't turn it in.)* Let $F = \{0, 1\}$. Find n, and a subspace $C \subseteq F^n$ such that $\dim(C) > \frac{n}{2}$, and for all i, $e_i \notin C$, and for all $i \neq j$, $e_i + e_j \notin C$. [This shows it is possible to transmit a message, and correct for at most one error, by transmitting *less* than twice your message.]

4. Let $T : U \to V$ be a function. Show that $T \in \mathcal{L}(U, V)$ if and only if $\forall u, v \in U$, and $\alpha \in F$, $T(u + \alpha v) = T(u) + \alpha T(v)$.

5. Suppose $\dim(V) = 1$, and $T \in \mathcal{L}(V, V)$. Show that $\exists \alpha \in F$ such that $T(v) = \alpha v$ for all $v \in V$.

6. (a) Suppose V is a vector space over $\mathbb{R}$, and $T : V \to V$ is such that $T(u + v) = T(u) + T(v)$ for all $u, v \in V$. Show that for all $q \in \mathbb{Q}$, and $v \in V$, we have $T(qv) = qT(v)$. [Surprisingly, we need not have $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$. The existence of such functions however involves the axiom of choice.]
   (b) Suppose now $V = \mathbb{R}$, and $T : \mathbb{R} \to \mathbb{R}$ satisfies $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$. If further T is continuous, show that $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{R}$.

7. Given an example of a field F, two (finite dimensional) vector spaces U, V over F, and two linear transformations $S \in \mathcal{L}(U, V)$, $T \in \mathcal{L}(V, U)$ such that $ST = I$, but $TS \neq I$. [Note $ST \in \mathcal{L}(V, V)$, so when we say $ST = I$, we really mean ST equals the identity element in $\mathcal{L}(V, V)$ (sometimes denoted by I_V, or 1_V). Similarly for $TS = I$, we mean $TS = I_U = 1_U$.]