Homework Assignment 3

Assigned Fri 01/21. Due Fri 01/21.

1. Determine whether B is a basis of V over F. Justify your answer.
   (a) $B = \left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 5 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 4 \\ -3 \end{smallmatrix}\right) \right\}$; $V = \mathbb{R}^3$; $F = \mathbb{R}$.
   (b) $B = \{1 + 2x + x^2, 1 + x, 1\}$, $F = \mathbb{R}$, $V = P_2(\mathbb{R})$ (polynomials of degree $\leqslant 2$).
   (c) $B = \{x^2, x + 1\}$; $V = P_2(F)$; $F = \{0, 1\}$.
   (d) $B = \left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 2 \\ -5 \end{smallmatrix}\right) \right\}$; $V = \left\{ x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$; $F = \mathbb{R}$.

2. The conclusion of both subparts below will follow directly from a general theorem we will prove later. However it's worth while doing them out explicitly by hand at least once…
   (a) Suppose u_1, u_2 are *any* two linearly independent vectors in $\mathbb{R}^2$, then show (by direct computation) that $\operatorname{span}\{u_1, u_2\} = \mathbb{R}^2$.
   (b) Let $V = \mathbb{R}^3$, and $U \subseteq \mathbb{R}^3$ be the plane $x_1 + x_2 + x_3 = 0$. Show (by direct computation) that if u_1, u_2 are *any* two linearly independent vectors in U, then $U = \operatorname{span}\{u_1, u_2\}$.

3. Let V be a vector space over a field F.
   (a) Suppose U and W are two subspaces of V. Are $U \cup W$ or $U \cap W$ always vector spaces? If yes, prove it. If no, furnish a counter example.
   (b) Define $U + W = \{u + w \mid u \in U, w \in W\}$. If U, W are subspaces of V, then show that $U + W$ is also a subspace of V.
   (c) If $V = \mathbb{R}^3$, $F = \mathbb{R}$, $U = \{ \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \mid x_1 + x_2 = 0\}$, and $W = \{ \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \mid x_1 + x_2 + x_3 = 0\}$, then compute $U + W$.

4. Let $L^2[0,1] = \{f \mid f : [0,1] \to \mathbb{R} \ni \int_0^1 f(x)^2 \, dx < \infty\}$. Define addition and scalar multiplication as you would for functions. Prove that $L^2[0,1]$ a vector space over $\mathbb{R}$. [Note $f : [0,1] \to \mathbb{R}$ means "f is a real valued function defined on $[0,1]$". Thus $L^2[0,1]$ is the set of all real valued functions f defined on $[0,1]$ such that $\int_0^1 f(x)^2 \, dx < \infty$.]

5. Here's another proof showing that the dimension of a vector space is well defined.
   (a) Let F be any field, $m < n \in \mathbb{N}$, and $\alpha_{ij} \in F$ be given. Show that there exists $x_1, x_2, \ldots, x_n \in F$ *not all* 0 such that

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= 0 \\ &\vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n &= 0. \end{aligned}$$

   [HINT: In words this problem says that any system of homogeneous linear equations has a non-zero solution, *provided* you have more variables than equations. The hint is to use induction. But don't get carried away and use some sort of fancy double induction trick on m and n. You can do this directly with induction on one of the variables.]
   (b) Suppose now V is a vector space over F, $m < n \in \mathbb{N}$, and $V = \operatorname{span}\{u_1, \ldots, u_m\}$. Show (using the previous subpart) that any subset of n vectors in V must be linearly dependent. [HINT: Let $v_1, \ldots, v_n$ be n vectors in V, and express each v_j as a linear combination $\sum_i \alpha_{ij} u_i$. Now somehow reduce linear dependence of v_j's to solving equations like in the previous subpart. Of course, as we've seen in class, this subpart immediately implies that any two (finite) basis in a vector space have the same number of elements.]
   (c) The statements in parts (a) and (b) above are really equivalent. Above you should have shown that part (a) implies part (b). Now do the converse: Namely, assuming the result of part (b) above, show that the result in part (a) is true.