Homework Assignment 2

Assigned Fri 01/14. Due Fri 01/21.

When unspecified, always assume F is a field, and V is a vector space over F.

1. Show that $\frac{-1\pm\sqrt{3}i}{2}$ is a cube root of 1.

2. For all $v \in V$, and $\alpha \in F$, show that $\alpha v = 0$ if and only if $\alpha = 0$, or $v = 0$.

3. For this question, $F = \mathbb{R}$, $V \subseteq \mathbb{R}^2$ is the specified subset, and addition and scalar multiplication are defined as the respective operation for $\mathbb{R}^2$.
   (a) Let $\alpha \in \mathbb{R}$, $V = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = \alpha\}$. For what values of α is V a vector space?
   (b) Let $V = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mid x_1, x_2 \in \mathbb{R}, x_2 = x_1^2\}$. Is V a vector space? Justify.
   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. $V = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mid x_1, x_2 \in \mathbb{R}, x_2 = f(x_1)\}$. Show that V is a vector space, if and only if $\exists \alpha \in \mathbb{R}$, such that $f(x) = \alpha x$ for all $x \in \mathbb{R}$.

4. Let F we a field. We define $P(F)$ to be the set of all polynomials over F. That is

$$P(F) = \{f \bigm| \exists n \in \mathbb{N}, \text{ and } a_0, \dots, a_n \in F \ni f(x) = a_0 + a_1 x + \cdots + a_n x^n, \ \forall x \in F\}.$$

   We define vector addition, and scalar multiplication as we did for functions (i.e. $f + g$ is the function such that for all $x \in F$, $(f + g)(x) = f(x) + g(x)$).
   (a) Show that $P(F)$ is a vector space.
   (b) Suppose now that $F = \mathbb{R}$ or $F = \mathbb{C}$. Let $f \in P(F)$. Show that there exists a unique $n \in \mathbb{N}$, and unique $a_0, \dots, a_n \in F$ with $a_n \neq 0$, such that $f(x) = a_0 + a_1 x + \cdots a_n x^n$ for all $x \in F$. This unique n is called the degree of the polynomial f. [Note: What you have to prove here is to suppose $f(x) = \sum_0^m a_i x^i = \sum_0^n b_i x^i$, with $a_m, b_n \neq 0$, and show that this necessarily means $m = n$, and $a_i = b_i$ for all i.]
   (c) Show that the previous problem is *false* if we don't assume $F = \mathbb{R}$ or $F = \mathbb{C}$.
   (d) Let $F = \mathbb{R}$ or $F = \mathbb{C}$. Let U be all elements of $P(F)$ with degree *exactly* n. With addition and scalar multiplication defined as in $P(F)$, is U a vector space? Provide a proof, or counter example.
   (e) Let $F = \mathbb{R}$ or $F = \mathbb{C}$, and let $P_n(F)$ be all elements of $P(F)$ with degree *less than or equal to* n. Is $P_n(F)$ a vector space? Provide a proof, or counter example.

5. *(Quarternions)* As mentioned today, the cross product on $\mathbb{R}^3$ really arises from a 'vector multiplication' on $\mathbb{R}^4$. This problem describes that structure: Let i, j, k be 'numbers', with a multiplication defined such that $i^2 = j^2 = k^2 = ijk = -1$.
   (a) Assuming that $\{\pm 1, \pm i, \pm j, \pm k\}$ satisfy the multiplicative field axioms except commutativity, write down a multiplication table for them.
   (b) Now for $x_i, y_i \in \mathbb{R}$, formally define

$$(x_1 + ix_2 + jx_3 + kx_4)(y_1 + iy_2 + jy_3 + ky_4)$$

   by using the distributive law, and your rules for multiplying i, j, k from the previous part. Also define addition component wise:

$$(x_1 + ix_2 + jx_3 + kx_4) + (y_1 + iy_2 + jy_3 + ky_4) = (x_1 + y_1) + (x_2 + y_2)i + (x_3 + y_3)j + (x_4 + y_4)k$$

   Show that this addition and multiplication above satisfy all the field axioms, *except* commutativity of multiplication. [Most of the axioms are straightforward enough to verify, and torturous to write down. Only explicitly write down the proof that multiplicative inverses exist. Verify the remaining for yourself, but don't turn it in.]