Homework Assignment 11
Assigned Fri 04/22. Due Fri 04/29.

1. Let \( V \) be an inner product space.
   (a) If \( U \subseteq V \) is a subspace, show that \( U \subseteq (U^\perp)^\perp \).
   (b) If \( \dim(V) < \infty \), show that \( U = (U^\perp)^\perp \).

2. Suppose we want to find a line \( y = c_0 + c_1x \) that ‘best fits’ the points \((0,0), (1,1), (1,2)\).
   (a) If indeed there was a line that passed through the three points above, show that the coefficients
      \( c_0 \) and \( c_1 \) satisfy \( A(\frac{c_0}{c_1}) = \vec{b} \), where \( \vec{b} = \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right) \), and \( A = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \).
   (b) With \( A \) and \( \vec{b} \) as above, show that the system \( A\vec{x} = \vec{b} \) has no solutions.
   (c) Let \( \left( \frac{c_0}{c_1} \right) \) be the least square solution to \( A\vec{x} = \vec{b} \). Sketch the line \( y = c_0 + c_1x \) and the points \((0,0), (1,1), (1,2)\) on the same graph.

3. Let \( U, V \) be two finite dimensional inner product spaces, with inner products \( \langle \cdot, \cdot \rangle_U \) and \( \langle \cdot, \cdot \rangle_V \) respectively.
   (a) If \( T \in \mathcal{L}(U, V) \), show that there exists a unique \( T^* \in \mathcal{L}(V, U) \) such that for all \( u \in U, v \in V \),
      \( \langle Tu, v \rangle_V = \langle u, T^*v \rangle_U \).
   (b) Let \( T \in \mathcal{L}(U, V) \). Show that \( \ker(T^*) = \text{im}(T)^\perp \), and \( \text{im}(T^*) = \ker(T)^\perp \).
   (c) If \( T \in \mathcal{L}(U, V) \) is injective, show that \( T^*T \in \mathcal{L}(U, U) \) is invertible. Is \( TT^* \in \mathcal{L}(V, V) \) invertible?
   (d) Let \( W \subseteq \mathbb{R}^n \) be a subspace, and \( \{w_1, \ldots, w_m\} \subseteq W \) be a basis of \( W \). Let \( A \in \text{Mat}(n, m, \mathbb{R}) \) be the matrix with \( w_1, \ldots, w_m \) as columns. Show that \( P = A(A^tA)^{-1}A^t \) is the orthogonal projection of \( \mathbb{R}^n \) onto \( W \). [You should also justify why \( A^tA \) is invertible. Note that given a basis of a subspace, this problem gives you an explicit formula for the orthogonal projection. However, computationally, this formula is about as much ‘work’ as using Gram-Schmidt first, and using the formula from class.]

4. Suppose \( \dim(V) < \infty \) is an inner product space over \( F \), and \( T \in \mathcal{L}(V, V) \). Suppose further there exists an orthonormal basis of \( V \) consisting of eigenvectors of \( T \).
   (a) If \( F = \mathbb{C} \), show that \( T^*T = TT^* \).
   (b) If \( F = \mathbb{R} \), show that \( T^* = T \).

5. Let \( F = \mathbb{R} \), and \( V \) be an inner-product space. Recall \( T \in \mathcal{L}(V, V) \) is orthogonal if \( \langle Tu, Tv \rangle = \langle u, v \rangle \) for all \( u, v \in V \).
   (a) Show that \( T \) is orthogonal if and only if \( \|Tu\| = \|u\| \) for all \( u \in V \).
   (b) Show that \( T \) is orthogonal if and only if \( T^* = T^{-1} \).
   (c) Let \( V = \mathbb{R}^2 \), \( T \in \mathcal{L}(V, V) \) be orthogonal, and \( B = \{e_1, e_2\} \) be the standard basis. Show that there exists \( \theta \in [0, 2\pi) \) such that \( \mathcal{M}_B(T) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \) or \( \mathcal{M}_B(T) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \).
      [Elementary trigonometry now shows that the first matrix above is a rotation through angle \( \theta \), and the second is a reflection about line \( y = \tan(\frac{\theta}{2})x \). Thus all orthogonal transformations on \( \mathbb{R}^2 \) are either rotations or reflections. The same is true in higher dimensions (and you know enough to prove this).]

6. Let \( A \in \text{Mat}(n, n, \mathbb{C}) \). We define \( e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \). Show that \( \det(e^A) = e^{\text{tr}(A)} \). [Note: An elementary application of the comparison test will quickly show that the series \( \sum_{k=0}^{\infty} \frac{1}{k!} A^k \) converges (absolutely) for any matrix \( A \). For this question, you needn’t worry about issues of convergence. The hint is as follows: First check this for diagonal matrices. Next compute what \( e^{P^{-1}AP} \) is in terms of \( P \) and \( e^A \). Now if your matrix \( A \) was diagonalizable, you should have a quick proof. The proof of the general case will follow using the same
idea and question 3 from the previous homework. As a general remark – $e^A$ is called the matrix exponential, and is extremely useful in solving ODE’s. Proving this question using the ‘expansion by minors’ formula for the determinant is doomed to failure, but the decomposition of $V$ into (generalized) eigenspaces quickly gives the desired result.]