INVESTIGATIONS OF THE DYNAMICS OF MODELS OF HEAT TRANSFER AND CLUSTERING

by

Truong-Son Van
(Vietnamese: Văn Phụng Trường Sơn)

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Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA

Doctoral thesis committee

Robert L. Pego (CMU, co-chair)
Gautam Iyer (CMU, co-chair)
Noel J. Walkington (CMU)
Jean-Luc Thiffeault (UW-Madison)

Contents

Abstract

We investigate two models that are widely used in physics and engineering. Our main goal is to study conjectures made by physicists with full mathematical rigor.

The first model we investigate is the advection-diffusion equation. Here, we address the problem of optimizing heat transfer via an incompressible fluid in a bounded domain. We use techniques in probability theory to get bounds for the heat transfer rate. Asymptotically, we obtain matching lower and upper bounds (up to a logarithmic factor) over a class of velocity field of the fluid that satisfies an energy-like constraint. This gives a rigorous proof for a result by Marcotte et. al. (SIAM Appl. Math '18). We also give an upper bound for for the problem under an enstrophy-like constraint.

The second model is the coagulation-fragmentation equation, which models the evolution ofthe density particle sizes in a system where particles can split and merge. Depending on the coagulation and fragmentation kernels, solutions of the system will behave differently. Here, we address two problems. The first problem concerns the well-posedness of mass-conserving solutions when the coagulation kernel is multiplicative and the fragmentation kernel constant. This belongs to a so-called critical case, where existence of mass-conserving solutions depends on how large the system is initially. Here, we develop a new technique by studying properties of the viscosity solutions of a corresponding singular Hamilton-Jacobi equation to deduce information about the solutions to the coagulation-fragmentation equation. Using this technique, we moved one step closer to resolving a long-standing conjecture in the field.

Still under the umbrella of the coagulation-fragmentation equation, we study the dynamics of the solution when the coagulation kernel is multiplicative and the fragmentation kernel additive and small. The problem we are concerned here resembles singular perturbation problems in PDEs. Letting the fragmentation kernel vanish, in the limit, one expects that the solutions tend to the so-called Flory solution of the pure multiplicative coagulation equation, where part of the total mass escapes to infinity. We study how the lost mass behaves. Our proposed idea is based on the study of a nonlinear backward parabolic equation, result from the Bernstein transform of the equation, and a detailed study of the tail behavior of the Flory solution of the pure coagulation equation with multiplicative kernel. With this idea, we made some progress towards resolving a prediction by Ben-Naim and Krapivsky (Phys. Rev. E 83, 061102, 2011).

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Overview

Essentially, mathematics becomes "applied" when it is used to solve real-world problems "neither seeking nor avoiding mathematical difficulties".

Lord Rayleigh

This thesis follows the spirit of the quote by Lord Rayleigh as it is an attempt to provide completely mathematically rigorous analyses of whatever real-world problems we could get our hands on. We classify these studies into two overarching themes: (1) Optimizing heat transfer, and (2) Understanding clusterization.

Optimizing heat transfer

The first theme, optimizing heat transfer, constitutes the first part of this thesis. This has a long history as it is one of the fundamental problems in engineering and physics. One of the most studied phenomena in heat transfer is the Rayleigh-Bernard convection, in which convection rolls naturally arise in thin liquid layers due to the difference in temperature of the boundaries of the liquid. These convection rolls serve as natural heat-transport enhancers within the liquid itself. The mathematical literature about the Rayleigh-Bernard convection is vast. Even though much still remains to be done, a particular method that has proven to be very fruitful is the so-called *background field method*, pioneered by Constantin and Doering [DC96].

A related and crucial question is, "Can we beat nature?". That is, can we build a fluid velocity field that transfer heat faster than the natural Rayleigh-Bernard convection rolls? Answers to this is fundamental to building optimal radiators/air-conditioners and heat exchangers for modern industries such as nuclear plants and computing processors. Despite a lot of work in the engineering community and numerical studies, rigorous mathematical works to find optimal ways to transport heat remain largely minimal until recently. To my knowledge, the first completely mathematically rigorous works in finding an optimal fluid flow to transport heat wall-to-wall are the works of Doering and coauthors (see, e.g., [DT19a]).

One of the simplest ways to model heat exchangers in a domain Ω is by using the advection-diffusion equation of heat concentration $c(x,t)$

$$\text{(AD)} \qquad \begin{cases} \dfrac{\partial c}{\partial t} = u \cdot \nabla c + \kappa \Delta c & \text{in } \Omega\,, \\ c = \nabla c \cdot n = 0 & \text{on } \partial\Omega \end{cases}$$

where u is an incompressible fluid flow and $\kappa > 0$ is the thermal diffusivity. The objective is to find the best u that satisfies some energy constraint so that $c(x,t)$ converges to 0 as fast as possible as $t \to \infty$ in some sense. From my current understanding, mathematically, this is a notoriously hard question.

Inspired by the heuristics and numerics of Marcotte, Doering, Thiffeault and Young [Mar+18a], we consider the problem of average exit time $T(x)$ of a Brownian motion with drift u starting at $x \in \Omega = (0,1)^2$. By the Dynkin formula, T satisfies

$$\begin{cases} -\dfrac{1}{2}\Delta T + u \cdot \nabla T = 1 & \text{in } \Omega\,, \\ T(\cdot, 1) = \dfrac{\partial T(\cdot, 0)}{\partial x_2} = 0\,. \end{cases}$$

Here, u obeys the energy condition $\|u\|_{L^p} \leqslant \mathrm{Pe}$, where Pe is the so-called Péclet number, which measures the relative strength between advection and diffusion. This problem is more tractable and still provides valuable insight into the study of (AD). We obtained the following theorem

THEOREM. *For $p \in [1,2]$ and large enough* Pe*, the lowest temperature that one could achieve by transferring heat via an incompressible flow that satisfies the above energy constraint is approximately bounded above by $O(\ln(\mathrm{Pe})/\mathrm{Pe})$. When $p = \infty$, the lowest temperature is bounded below by $O(1/\mathrm{Pe})$.*

The proof of this theorem involves two separate analyses, one for the upper bound and one for the lower bound. For the upper bound, our method of proof is a combination of rescaling the problem to obtain a degenerate advection-diffusion equation, a lot of careful large-deviation-type estimates and sub/super-solution barriers coming from Friedlin-Wentzell averaging problem. For the lower bound, we made a simple observation that utilizes the solution to the Eikonal equation to construct a subsolution that satisfies the bound. Our result confirms the prediction by Marcotte et al. [Mar+18a] and is a joint work with Gautam Iyer [IV21].

Understanding clusterization

The second part of this thesis is devoted to the study of clusterization of particles. The simplest equation that models this phenomenon is the so-called coagulation-fragmentaiton equation:

$$\partial_t \rho = Q_c(\rho) + Q_f(\rho)\,,$$

where $\rho(s,t)$ denotes the density of particles of size s at time t,

$$Q_c(\rho(s,t)) = \frac{1}{2}\int_0^s K(y, s-y)\rho(y,t)\rho(s-y,t)\,dy - c(s,t)\int_0^\infty K(s,y)\rho(y,t)\,dy\,,$$

and

$$Q_f(\rho(s,t)) = -\frac{1}{2}\rho(s,t)\int_0^s F(s-y,y)\,dy + \int_0^\infty F(s,y)\rho(y+s,t)\,dy$$

are called coagulation term and fragmentation term, respectively. The symmetric functions $K, F : [0,\infty)^2 \to [0,\infty)$ are called *coagulation kernel* and *fragmentation kernel*, respectively. They are physical measurements of the rates of binary merging and binary splitting of particles.

Despite being discovered more than a century ago [Smo16b], this equation is still not understood as there are a lot of interesting phenomena that come with it, depending on the kernels that may one consider. A particularly interesting phenomenon of the coagulation-fragmentation is that given the right conditions, the solution, while still physical, does not conserve mass at all time. There are two ways that this could happen. One comes from the formation of particles of infinite size; the other comes from the formation of particles of size zero, both in finite time. The first, called *gelation*, happens when the coagulation is strong enough [Esc+03]. The latter, called *dust formation*, happens when the fragmentation is strong enough (see Bertoin [Ber06]). Typically, these phenomena happen depending on the relative strengths between the coagulation kernel and fragmentation kernel, not so much on the initial data. However, there are borderline cases where it is not very clear how solutions would behave and hence a more careful analysis needs to be done based on initial data. Both are very interesting and rich phenomena, and have been studied in various contexts.

Critical well-posedness. We [TV21] considered the following pair of kernels

$$K(x,y) = xy \quad \text{and} \quad F(x,y) = 1$$

called multiplicative coagulation and constant fragmentation, respectively. In the literature, this is an example of a pair of critical kernels where the well-posedness of mass-conserving solution depends on how large the initial mass is [Esc+03]. A long-standing conjecture for this particular pair of kernels is that if the initial total mass of the system is less than 1, solutions will conserve mass. Otherwise, if the initial total mass is greater than 1, there will be lost of mass to infinity [VZ89]. We made progress in resolving this conjecture by introducing a new point of view, studying well-posedness and regularity of viscosity solutions to a particular Hamilton-Jacobi equation resulting from the so-called Bernstein transform applied to the coaguation-fragmentation equation. In this work, our main contribution, among other important results, is the following.

Theorem. *Solutions to the multiplicative-constant coagulation-fragmentation equation conserve mass if the initial mass is less than* $1/2$ *and fail to conserve mass if the initial mass is larger than* 1*.*

At the time this thesis is being written, our well-posedness result improves the previous threshold from $1/(4\log 2)$ [Lau19a] to $1/2$ while our non-well-posedness result is the most general in the literature as it only requires minimal assumptions on the initial data. This is a joint work with Hung V. Tran [TV21].

Singular perturbation. In the direction of the dynamics of the coagulation-fragmentation equation, we considere the following pair of kernels

$$K(x,y) = xy \quad \text{and} \quad F(x,y) = \varepsilon(x+y)\,.$$

For $\varepsilon > 0$, da Costa showed that the coagulation-fragmentation equation is well-posed as solutions conserve mass for all time $t > 0$ [Cos95]. However, when $\varepsilon = 0$, gelation occurs and the is a loss of mass to infinity after some time T_{gel}. Therefore, as $\varepsilon \to 0$, there is a cluster of giant particles that escape to infinity. Our goal is to study the dynamics of these giant particles. By studying a nonlinear singular backward parabolic equation resulting from the Laplace transform of the coagulation-fragmentation equation and the sharp decay rate of the Flory solution (which is important by itself), we are very close to resolving the following conjecture

Conjecture. *For some finte time and small enough ε, the giant particles concentrate around a metastable manifold that depends on the lost mass of the so-called Flory solution to the pure coagulation equation ($\varepsilon = 0$).*

This conjecture is a variant of a prediction by Ben-Naim and Krapivsky [BK11], which was inspired by stochastic simulations. This is a joint work with Bob Pego [PV21].

Part I

Heat transfer rate via passive advection

CHAPTER 1

Introduction

Suppose we live in the real world with three space dimensions, $\mathbb{R}^3$. For a closed system with still incompressible medium that doesn't interact with exogenous sources, the first law of thermodynamics says

$$Q = \frac{dU}{dt}\,, \tag{I.1.0.1}$$

where Q is the heat transfer rate and U the internal energy. In 1822, Fourier published a study of heat flow called *Théorie analytique da la chaleur* (The Analytical Theory of Heat) [Fou09], in which he proposed the famous law that bears his name: "the heat flux resulting from thermal conduction is proportional to the magnitude of the temperature gradient and opposite to it in sign". In mathematical terms,

$$q = -k\nabla\theta\,, \tag{I.1.0.2}$$

where q is the heat flux and θ the temperature. The constant k is called the *thermal conductivity* and changes depending on the medium under consideration. It is then a standard exercise in a physics class to deduce the well-known heat equation, which models how heat is transfered in a still incompressible medium [LL20]

$$c\rho\frac{\partial\theta}{\partial t} = k\Delta\theta\,,$$

where c and ρ are the heat capacity and density of the medium, respectively. Normalizing this equation appropriately, we arrive at a more friendly looking expression

$$\frac{\partial\theta}{\partial t} = \frac{1}{2}\Delta\theta\,. \tag{I.1.0.3}$$

This equation is one of the most basic partial differential equations. Humans seem to have exhausted what there is to know about it (for basic introduction, we refer the reader to the classics [Eva10; Str08a] and references therein).

The situation becomes more interesting when the medium is *not still* and there is convection (modeled by a velocity field $v \in \mathbb{R}^3$). One can still deduce that heat transfer can be modeled by the advection-diffusion equation [LL20]

$$\partial_t\theta = v \cdot \nabla\theta + \frac{1}{2}\Delta\theta\,. \tag{I.1.0.4}$$

However, the behavior of θ is far from understood with different kinds of velocity fields. In this thesis, we will look at this equation through the lens of stochastic analysis, which seems to be tremendously useful to heuristically read off information about θ.

I.1.1. Optimizing heat transfer

A heat exchanger is a system used to transfer heat between a fluid and a heat source or sink, for either heating or cooling. These are used for both heating and cooling processes and have a broad range applications including combustion engines, sewage treatment, nuclear power plants and cooling CPU's in personal computers [WBZ92; QM02; VP14; She+19; AK18; Mar+18b; Wan+18; DT19b; LL20].

Suppose the fluid is stirred from the outside. Let $v = v(x,t)$ be its velocity field. The temperature θ of the fluid in the heat exchanger evolves according to the advection diffusion equation (I.1.0.4) in a domain $\Omega \subseteq \mathbb{R}^d$ that is the region occupied by the fluid. Throughout this thesis we will assume the fluid is incompressible and doesn't flow through the container walls. That is, we require hence require

$$\nabla \cdot v = 0 \quad \text{in } \Omega\,, \qquad \text{and} \qquad v \cdot \hat{n} = 0 \quad \text{on } \partial\Omega\,. \tag{I.1.1.1}$$

Some portion of the boundary of Ω may be insulated, and some portion may be connected to a heat source/sink maintained at a constant temperature. Denoting these pieces by $\partial_N\Omega$ and $\partial_D\Omega$ respectively, and normalizing so that the temperature of the heat source/sink is 0, we study (I.1.0.4) with mixed Dirichlet/Neumann boundary conditions

$$\partial_{\hat{n}}\theta = 0 \quad \text{on } \partial_N\Omega\,, \qquad \text{and} \qquad \theta = 0 \quad \text{on } \partial_D\Omega\,.$$

A problem of practical interest is to minimize the temperature under a constraint on the stirring velocity field. Note, here we assume (I.1.0.4) is a passive scalar equation – the velocity field v is prescribed and is not coupled to the temperature profile. The active scalar case entails coupling v to θ via the Boussinesq system and leads to Rayleigh–Bénard convection which has been extensively studied [Ray16; SG88; Kad01; DOR06].

In order to simplify matters, we assume v is time independent, and assume the initial temperature θ_0 is identically 1. In this case we note that

$$T \stackrel{\text{def}}{=} \int_0^\infty \theta(x,t)\,dt$$

satisfies the Poisson problem

$$-\frac{1}{2}\Delta T + v \cdot \nabla T = 1\,, \tag{I.1.1.2}$$

in Ω, with boundary conditions

$$T = 0 \quad \text{on } \partial_D\Omega\,, \qquad \text{and} \qquad \partial_{\hat{n}}T = 0 \quad \text{on } \partial_N\Omega\,. \tag{I.1.1.3}$$

This part of the thesis is devoted to study how one can minimize T under various constraints on the advecting velocity field v.

I.1.2. Some mathematical tools

The mathematical tools in this part are mainly probabilistic. We restate without proofs here some of the standard results that underlie our approach. It is unfortunate that the language of stochastic analysis requires a great deal of technicality just to make sense of certain "basic" ideas. To communicate the general picture without bombarding the reader with pages of technical details, we will be somewhat informal in this chapter. To fully appreciate the beauty of stochastic analysis, the one can consult standard books by Øksendal, Karatzas and Shreve [Øks03a; KS91] for rigorous treatments of these results. The book by Feller [Fel71], though a bit dated, is perhaps the most enjoyable treatment that is very well written and full of insights from a true master of the topic.

Given a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Consider the following stochastic differential equation in $\mathbb{R}^d$

$$\begin{cases} dZ_t &= v(Z_t)\,dt + \sigma(Z_t)dB_t\,, \\ Z_0 &= X\,, \end{cases} \tag{I.1.2.1}$$

where $X : \Omega \to \mathbb{R}^d$ is a random variable, $v : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are nice functions so that (I.1.2.1) has a unique t-continuous solution such that

$$\boldsymbol{E}\int_0^\infty |Z_t|^2\,dt < \infty\,.$$

Let B_t be a standard d-dimensional Brownian motion and $\mathcal{F}_t = \sigma(B_s\,; 0 \leqslant s \leqslant t)$. Z_t is called an Itô diffusion and enjoy the strong Markov property, i.e., for any bounded Borel function on $\mathbb{R}^d$ and stopping time (with respect to $\mathcal{F}_t$) $\tau < \infty$ a.s., it is true that

$$\boldsymbol{E}^x\Big(f(Z_{\tau+h}) \mid \mathcal{F}_\tau\Big) = \boldsymbol{E}^{X_\tau} f(Z_h)\,,$$

were $\mathcal{F}_\tau = \sigma\Big(B_{\tau\wedge s}\,; s \geqslant 0\Big)$.

Remark I.1.2.1. Note that Z_t would not be an Itô diffusion but only an Itô process if v or σ explicitly depends on time as well.

There are advantages of being an Itô diffusion instead of merely an Itô process. One of the most useful advantages is that the generator of an Itô diffusion a differential operator. In particular, define

$$Af(z) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{\boldsymbol{E}^z f(Z_t) - f(x)}{t}$$

over appropriate set of functions $f : \mathbb{R}^d \to \mathbb{R}$. The operator A is called the generator of Z_t and we have the following formula

$$Af(z) = \sum_{i=1}^d b_i(z)\partial_i f(z) + \frac{1}{2}\sum_{i,j=1}^d (\sigma(x)\sigma^T(x))\partial_{ij}^2 f(z)\,.$$

From here, one could rephrase a lot of different problems in elliptic and parabolic PDEs as problems in stochastic analysis. For example, consider a bounded domain $D \subset \mathbb{R}^d$ and let $\tau = \inf\{t \geqslant 0 \mid Z_t \notin D\}$ be the first exit time of Z_t from D. Then, for appropriate f, the function

$$u(z) \overset{\text{def}}{=} \boldsymbol{E}^z \int_0^\tau f(Z_t)\,dt$$

solves the following PDE

$$\begin{cases} Au = -f\,, & x \in D\,,\\ u = 0\,, & x \in \partial D\,. \end{cases}$$

In particular, in this work, we will be concerned with $f \equiv 1$, which reduces our problem to studying the expected exit time $\boldsymbol{E}^z\tau$ of Z_t. Further discussions (with details) about generators of Itô diffusions and their relation to PDEs could be found in chapters 8 and 9 of the book [Øks03a].

A crucial tool that we will need from stochastic analysis is the Girsanov theorem. This theorem is a stochastic version of coordinate change with respect to certain flows that are different from the original flow that one is considering. It is particularly useful in the study of large deviation. We state here without proof a version of this theorem.

Theorem I.1.2.2 (Girsanov theorem [Øks03a, Theorem 8.6.8]). *Let $T > 0$, Z_t be an Itô diffusion of the form (I.1.2.1) with $X = x \in \mathbb{R}^d$ and Y_t be an Itô process of the form*

$$\begin{cases} dY_t = (\gamma(t,\omega) + v(Y_t))\,dt + \sigma(Y_t)\,dB_t\,,\\ Y_0 = x\,. \end{cases}$$

Suppose there exists a process $k(t,\omega)$ such that

$$\sigma(Y_t)k(t,\omega) = \gamma(t,\omega)\,.$$

For $t \leqslant T$, let

$$M_t = \exp\Big(-\int_0^t k(s,\omega)\,dB_s - \frac{1}{2}\int_0^t k^2(s,\omega)\,ds\Big)\,,$$
$$dQ(\omega) \overset{\text{def}}{=} M_T(\omega)\,d\boldsymbol{P}(\omega)\,,$$

and

$$\hat{B}_t \overset{\text{def}}{=} \int_0^t k(s,\omega)\,ds + B_t\,.$$

For appropriate conditions on v, γ and k, we have

$$dY_t = v(Y_t)\,dt + \sigma(Y_t)\,d\hat{B}_t\,,$$

and $\boldsymbol{Q}$-law of Y_t is the same as $\boldsymbol{P}$-law of Z_t for $0 \leqslant t \leqslant T$.

CHAPTER 2

Heat transfer rates

I.2.1. Introduction

In the recent paper [Mar+18b], the authors studied this minimization problem when $\Omega \subseteq \mathbb{R}^2$ is a disk of radius 1, and $\partial_N \Omega = \emptyset$. Given $p \in [1, \infty)$ and $\mathscr{U} > 0$, let $\mathcal{V}_{\mathscr{U}}^{k,p}$ be the set of all $W^{k,p}$ velocity fields satisfying (I.1.1.1) such that

$$\|v\|_{W^{k,p}(\Omega)} \leqslant \mathscr{U} \,, \tag{I.2.1.1}$$

and define

$$\mathcal{E}_q^{k,p}(\mathscr{U}) \stackrel{\text{def}}{=} \inf_{v \in \mathcal{V}_{\mathscr{U}}^{k,p}} \|T^v\|_{L^q} \,.$$

Here T^v is simply the solution to (I.1.1.2)–(I.1.1.3), and we introduced the superscript v to emphasize the dependence of T on v.

Physically when $k = 0$ and $p = 2$, the constraint (I.2.1.1) limits the kinetic energy of the ambient fluid. If the domain Ω has an associated length scale of order 1, the quantity $\mathscr{U}$ is the *Péclet number* — a non-dimensional ratio measuring the relative strength of the advection to the diffusion. When the Péclet number is sufficiently large, the authors of [Mar+18b] use matched asymptotics to show

$$\mathcal{E}_1^{0,2}(\mathscr{U}) \leqslant O\Big(\frac{1}{\mathscr{U}}\Big) \,, \tag{I.2.1.2}$$

and support their results with numerics.

In this chapter we revisit this problem and aim to provide mathematically rigorous proofs of the bounds in [Mar+18b]. Making matched asymptotics rigorous arises in many situations and has been extensively studied (see for instance [BLP78; Kus84; Ngu89; Eva90; All92; PS08]). In this situation, however, the flow considered in [Mar+18b] leads to a degenerate homogenization problem, for which one can not use standard techniques. Instead we reformulate the problem probabilistically and use asymmetric large deviations estimates to handle the degenerate diffusivity.

To simplify the proofs, we study the problem in a horizontal strip instead of the disk. For boundary conditions we cool the top of the strip, insulate the bottom, and impose 2-periodic boundary conditions in the horizontal direction. To prove the upper bound $\mathcal{E}_q^{0,p}(\mathscr{U})$ we only need to find a velocity field $v \in \mathcal{V}_{\mathscr{U}}^{0,p}$ for which $\|T^v\|_{L^q} \leqslant O(1/\mathscr{U})$. A natural first guess would be to choose a velocity field that forms many tall and thin convection rolls, with height $O(1)$, and width / amplitude that depend on the Péclet number. This, however, turns out

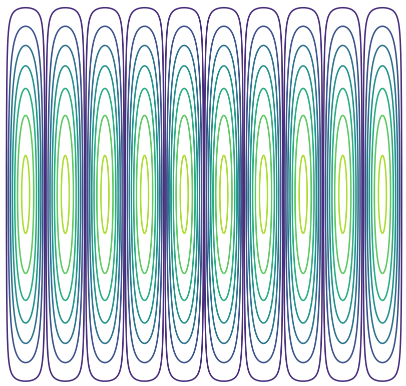

FIGURE I.2.1.1. Tall and thin convection rolls

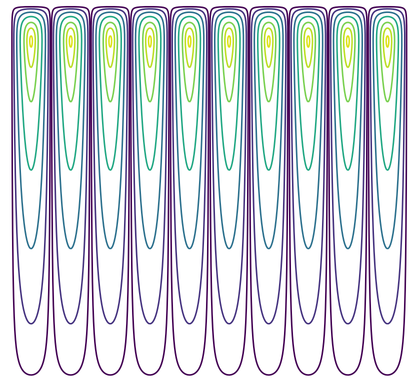

FIGURE I.2.1.2. Skewed tall and thin convection rolls.

to be suboptimal, and yields a bound that is worse than (I.2.1.2). To obtain the bound (I.2.1.2) one needs to consider tall and thin convection rolls whose center is very close to the top of the strip. This is the analogue of the velocity fields used in [Mar+18b], and is shown in I.2.1.2.

To formulate our result precisely, let $S = \mathbb{R} \times (0,1) \subseteq \mathbb{R}^2$ be an infinite horizontal strip and $\partial_D S = \mathbb{R} \times \{1\}$ be the top boundary (where we impose homogeneous Dirichlet boundary conditions), and $\partial_N S = \mathbb{R} \times \{0\}$ the bottom boundary (where we impose homogeneous Neumann boundary conditions). We will impose 2-periodic boundary conditions in the horizontal direction and identify the function spaces $H^1(S)$ and $L^2(S)$ with $H^1(\Omega)$ and $L^2(\Omega)$, respectively, where $\Omega \stackrel{\text{def}}{=} (0,2) \times (0,1)$.

THEOREM I.2.1.1. *There exists a constant C such that for $q \in [1,\infty]$,*

$$\mathcal{E}_q^{0,\infty}(\mathscr{U}) \geqslant \frac{1}{C\mathscr{U}}\,. \tag{I.2.1.3}$$

Furthermore, for every $\mu > 0$, $p, q \in [1,\infty]$, we have

$$\begin{cases} \mathcal{E}_q^{0,p}(\mathscr{U}) \leqslant \dfrac{C \ln \mathscr{U}}{\mathscr{U}} & p \in [1,2)\,, \\ \mathcal{E}_q^{0,p}(\mathscr{U}) \leqslant \dfrac{C_\mu \ln \mathscr{U}}{\mathscr{U}^{\frac{2p}{3p-2}-\mu}} & p \in [2,\infty]\,, \end{cases} \tag{I.2.1.4}$$

whenever the Péclet number, $\mathscr{U}$, is sufficiently large.

For $p, q < \infty$, upper bound in (I.2.1.4) is suboptimal. Indeed, forthcoming work of Doering and Tobasco uses methods in [DT19b] to show that

$$\mathcal{E}_q^{0,p}(\mathscr{U}) \leqslant \frac{C}{\mathscr{U}} \qquad \text{for every } p, q \in [1,\infty)\,, \tag{I.2.1.5}$$

and some constant $C = C(p,q)$ and all sufficiently large $\mathscr{U}$. This is an improvement of (I.2.1.4) by a logarithmic factor for $p \in [1,2)$, and an arbitrarily small algebraic power for $p = 2$, and by a fixed algebraic power for $p \in (2,\infty)$. For $q = \infty$, however, the methods in [DT19b] do not work. In this case we believe that the logarithmic factor in (I.2.1.4) is necessary due to the presence of hyperbolic critical points, but we are presently unable to prove this.

We do not presently know how to prove any lower bound for $\mathcal{E}_q^{0,p}(\mathscr{U})$ when $p < \infty$. For $p = \infty$, however, we can use the Eikonal equation to obtain the lower bounded stated in (I.2.1.4) in general domains. We state this result next.

Proposition I.2.1.2. *Let $d \geqslant 2$, and $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Decompose the boundary as $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, with $\partial_D\Omega \neq \emptyset$. Then*

$$\mathcal{E}_q^{0,\infty}(\mathscr{U}) \geqslant \frac{1}{C\mathscr{U}} \qquad \text{for every } q \in [1,\infty]\,, \tag{I.2.1.6}$$

for some constant $C = C(\Omega)$, and all sufficiently large $\mathscr{U}$.

Remark. As we will see in the proof (specifically from inequality (I.2.2.2), below), the constant $C = C(q,\Omega)$ can be computed in terms of the L^q norm of the solution to the Eikonal equation in Ω.

Next we study the behavior of $\mathcal{E}_q^{1,p}(\mathscr{E})$ when $\mathscr{E}$ is large. Physically this corresponds to minimizing the L^q norm of the steady state temperature T under an *enstrophy* constraint on the stirring velocity field. In this case it turns out that using standard convection rolls (as shown in Figure I.2.1.1) yields a better upper bound on $\mathcal{E}_q^{1,p}(\mathscr{E})$ than the skewed tall and thin rolls (as shown in Figure I.2.1.2). We note, however, that we have no matching lower bound and the skewed tall and thin convection rolls may not provide the optimal upper bound. Indeed, the branched flows introduced recently by Doering and Tobasco [DT19b] may provide the optimal bound in the enstrophy constrained case. Unfortunately, due to their complicated geometry, they can not be analyzed by the techniques we use. The best bound we can obtain is as follows.

Proposition I.2.1.3. *For every $p,q \in [1,\infty]$, there exists a finite constant $C = C(q)$ such that*

$$\mathcal{E}_q^{1,p}(\mathscr{E}) \leqslant \frac{C|\ln \mathscr{E}|^{13}}{\mathscr{E}^{2/5}} \tag{I.2.1.7}$$

whenever $\mathscr{E}$ is sufficiently large. One velocity field that attains this upper bound uses convection rolls with height 1, width $\mathscr{E}^{-1/5}$ and amplitude $\mathscr{E}^{3/5}$ (see Figure I.2.1.1).

Even though there may be "non-convection roll" like flows that could improve the upper bound (I.2.1.7), heuristics show that the bound (I.2.1.7) is the best one can achieve amongst the class of all "convection roll" like flows. Moreover, for the tall and thin convection rolls used in proof of Proposition I.2.1.3 one has matching upper and lower bounds on $\|T^v\|_{L^\infty}$, up to a logarithmic factor. Since such convection rolls arise in the study of magma flow in the Earth's mantle and various other contexts [TS02; KJ03; GHZ11; YVL15; Ost17], the techniques used in the proof of Proposition I.2.1.3 may be useful in some of these situations.

For a lower bound, clearly $\mathcal{E}_q^{1,\infty}(\mathscr{E}) \geqslant \mathcal{E}_q^{0,\infty}(\mathscr{E})$, and hence by Proposition I.2.1.2 we have

$$\mathcal{E}_q^{1,\infty}(\mathscr{E}) \geqslant \frac{1}{C\mathscr{E}}\,, \qquad \text{for every } q \in [1,\infty]\,,$$

for all sufficiently large $\mathcal{E}$. We may be able to improve this by at most a logarithmic factor using a detailed analysis of the behavior near saddle points. However, as mentioned earlier, we do not know whether the upper bound (I.2.1.7) is optimal and we are unable to obtain a matching lower bound.

Plan of the chapter. In Section I.2.2 we prove the lower bounds in Theorem I.2.1.1 and Proposition I.2.1.2. In Section I.2.3, we use an elementary scaling argument to reduce Proposition I.2.1.3 to obtaining an upper bound on a degenerate cell problem (Proposition I.2.3.1). In Section I.2.4 we prove Proposition I.2.3.1 using probabilistic techniques, modulo two lemmas concerning exit from / the return to the boundary layer. These lemmas are proved in Sections I.2.5 and I.2.6. The proofs of these lemmas rely on certain large deviations estimates which relegated to Appendix 2.A. The proof of the upper bound in Theorem I.2.1.1 is similar to the proof of Proposition I.2.3.1, and is presented in Section I.2.7.

I.2.2. Lower bounds

In this section we prove the lower bound in Theorem I.2.1.1 and the generalized version in Proposition I.2.1.2. The main idea in the proof is to consider an incompressible flow that moves directly towards the cold boundary. Of course, this flow penetrates the boundary of the domain and so is not an element of $\mathcal{V}_{\mathscr{U}}^{0,\infty}$. However, it can still be used to build a sub-solution and prove the desired lower bound. Since the proof in a strip is short and explicit, we present it first.

Proof of the lower bound in Theorem I.2.1.1. Let $\underline{T}$ be the solution to

$$-\frac{1}{2}\partial_y^2 \underline{T} - \mathscr{U}\,\partial_y \underline{T} = 1$$

in the strip S with $\underline{T} = 0$ $\partial_D S$ and $\partial_y \underline{T} = 0$ on $\partial_N S$. Explicitly solving this yields

$$\underline{T}(y) = \frac{e^{-2\mathscr{U}}}{2\mathscr{U}^2}\Big(1 - e^{2\mathscr{U}(1-y)}\Big) + \frac{1-y}{\mathscr{U}} \tag{I.2.2.1}$$

and hence $\partial_y \underline{T} \leqslant 0$.

We now claim that for any velocity field v such that $v_2 \geqslant -\mathscr{U}$, the function $\underline{T}$ is a sub-solution to (I.1.1.2)–(I.1.1.3). Indeed,

$$-\frac{1}{2}\Delta \underline{T} + v \cdot \nabla \underline{T} = -\frac{1}{2}\partial_y^2 \underline{T} + v_2 \partial_y \underline{T} \leqslant -\frac{1}{2}\partial_y^2 \underline{T} + \mathscr{U}\partial_y \underline{T} = 1\,.$$

The last inequality above followed from the fact that $v_2 \geqslant -\mathscr{U}$ and $\partial_y \underline{T} \leqslant 0$.

Thus by the comparison principle, for every $v \in \mathcal{V}^{0,\infty}_{\mathscr{U}}$ we must have $0 \leqslant \underline{T} \leqslant T^v$. Hence $\|T^v\|_{L^q} \geqslant \|\underline{T}\|_{L^q}$ and computing $\|\underline{T}\|_{L^q}$ using (I.2.2.1) yields the lower bound in (I.2.1.4) as claimed. □

In general domains the sub-solution isn't as explicit and needs to be constructed using the Eikonal equation.

Proof of Proposition I.2.1.2. Let $v \in L^\infty(\Omega)$, and $T = T^v$ be the solution of (I.1.1.2). For any $\varepsilon > 0$ let $\tilde{T}^{\varepsilon,\lambda}$ be the solution to the following viscous Hamilton-Jacobi equation

$$\begin{cases} \lambda \tilde{T}^{\varepsilon,\lambda} - \varepsilon\Delta \tilde{T}^{\varepsilon,\lambda} + |\nabla \tilde{T}^{\varepsilon,\lambda}| = 1\,, & x \in \Omega\,,\\ \tilde{T}^{\varepsilon,\lambda} = 0\,, & x \in \partial\Omega\,. \end{cases}$$

Note that $\tilde{T}^{\varepsilon,\lambda} \geqslant 0$ as 0 is a subsolution to this equation. It is well known (see for instance [Cal18; Tra21]) that for every $\lambda > 0$, $\tilde{T}^{\varepsilon,\lambda}$ converges uniformly as $\varepsilon \to 0$ to the viscosity solution of the equation

$$\begin{cases} \lambda \tilde{T}^{0,\lambda} + |\nabla \tilde{T}^{0,\lambda}| = 1\,, & x \in \Omega\,,\\ \tilde{T}^{0,\lambda} = 0\,, & x \in \partial\Omega\,. \end{cases}$$

Now letting $\lambda \to 0$, $\tilde{T}^{0,\lambda}$ converges uniformly to the viscosity solution of the Eikonal equation

$$\begin{cases} |\nabla \tilde{T}^{0,0}| = 1\,, & x \in \Omega\,,\\ \tilde{T}^{0,0} = 0\,, & x \in \partial\Omega\,. \end{cases}$$

We claim that $\underline{T}^{\varepsilon,\lambda} \stackrel{\text{def}}{=} \varepsilon \tilde{T}^{\varepsilon,\lambda}$ is a sub-solution of (I.1.1.2) provided $\varepsilon \leqslant 1/\|v\|_{L^\infty}$. Indeed,

$$\begin{aligned} -\Delta \underline{T}^{\varepsilon,\lambda} + v \cdot \nabla \underline{T}^{\varepsilon,\lambda} &\leqslant -\Delta \underline{T}^{\varepsilon,\lambda} + \varepsilon \|v\|_{L^\infty} |\nabla \tilde{T}^{\varepsilon,\lambda}| \\ &\leqslant -\Delta \underline{T}^{\varepsilon,\lambda} + |\nabla \tilde{T}^{\varepsilon,\lambda}| + \frac{\lambda}{\varepsilon}\underline{T}^{\varepsilon,\lambda} = -\varepsilon\Delta \tilde{T}^{\varepsilon,\lambda} + |\nabla \tilde{T}^{\varepsilon,\lambda}| + \lambda \tilde{T}^{\varepsilon,\lambda} = 1\,. \end{aligned}$$

Since $\underline{T}^{\varepsilon,\lambda} = 0$ on $\partial\Omega$, and T^v is nonnegative, the minimum principle implies $\underline{T}^{\varepsilon,\lambda} \leqslant T^v$ in Ω. This immediately implies

$$\frac{1}{\varepsilon}\|T^v\|_{L^q} \geqslant \frac{1}{\varepsilon}\|\underline{T}^{\varepsilon,\lambda}\|_{L^q} \xrightarrow{\varepsilon\to 0} \|\tilde{T}^{0,\lambda}\|_{L^q} \xrightarrow{\lambda\to 0} \|\tilde{T}^{0,0}\|_{L^q} \,.$$

Thus when ε is sufficiently small we have

$$\|T^v\|_{L^q} \geqslant \frac{\varepsilon}{2}\|\tilde{T}^{0,0}\|_{L^q} \,.$$

Consequently, if $\|v\|_{L^\infty}$ is sufficiently large, we can choose $\varepsilon = \frac{1}{\|v\|_{L^\infty}}$ and obtain

$$\|T^v\|_{L^q} \geqslant \frac{1}{2\|v\|_{L^\infty}}\|\tilde{T}^{0,0}\|_{L^q} \,. \tag{I.2.2.2}$$

This immediately implies the bound (I.2.1.6) as claimed. $\square$

I.2.3. Upper bound for enstrophy constrained convection rolls

Our aim in this section is to prove Proposition I.2.1.3. First note that by doubling the domain and using symmetry and rescaling we can reduce the problem to proving (I.2.1.7) on the domain

$$S_2 \stackrel{\text{def}}{=} \mathbb{R} \times (-1,1)\,, \qquad \text{with} \qquad \partial_N S_2 = \emptyset\,, \qquad \partial_D S_2 = \mathbb{R} \times \{-1,1\}\,,$$

and only using velocity fields v for which

$$v_1(x_1,-x_2) = v_1(x_1,x_2) \qquad \text{and} \qquad v_2(x_1,-x_2) = -v_2(x_1,x_2)\,. \tag{I.2.3.1}$$

We will now prove the upper bound (I.2.1.7) by producing a velocity field v (depending on $\mathscr{E}$) such that we have

$$\|T^v\|_{L^\infty} \leqslant C|\ln \mathscr{E}|^{13}\Big(\frac{1}{\mathscr{E}}\Big)^{2/5}\,, \tag{I.2.3.2}$$

for all $\mathscr{E}$ sufficiently large. We do this by forming convection rolls with height 1, width ε and amplitude $A_\varepsilon/\varepsilon^2$ for some small ε and large A_ε (see Figure I.2.1.1). Moreover, as we will see shortly, ε and A_ε should be chosen according to

$$\frac{A_\varepsilon}{\varepsilon^3} = \mathscr{E}\,. \tag{I.2.3.3}$$

To construct v, consider a Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$H(x_1,-1) = H(x_1,1) = 0\,, \tag{I.2.3.4a}$$

$$H(x_1,-x_2) = -H(x_1,x_2)\,, \tag{I.2.3.4b}$$

$$H(x_1+2,x_2) = H(x_1,x_2)\,, \tag{I.2.3.4c}$$

for all $(x_1, x_2) \in \mathbb{R}^2$. To obtain convection rolls of width ε and height 1, we rescale the horizontal variable. Define

(I.2.3.5)
$$H^\varepsilon(x_1, x_2) = H\Big(\frac{x_1}{\varepsilon}, x_2\Big)\,, \qquad \text{and} \qquad v^\varepsilon = \frac{A_\varepsilon}{\varepsilon}\nabla^\perp H^\varepsilon = \frac{A_\varepsilon}{\varepsilon}\begin{pmatrix} \partial_2 H^\varepsilon \\ -\partial_1 H^\varepsilon \end{pmatrix}\,,$$

and let $T_\varepsilon = T^{v^\varepsilon}$. By uniqueness of solutions to (I.1.1.2) we see that T_ε satisfies $T_\varepsilon(x_1 + 2\varepsilon, x_2) = T_\varepsilon(x_1, x_2)$. Thus, it is natural to make the change of variables

$$y_1 = \frac{x_1}{\varepsilon}\,, \quad y_2 = x_2\,, \qquad \text{and} \qquad v = (v_1, v_2) = \nabla_y^\perp H\,. \tag{I.2.3.6}$$

In these coordinates we see that $T_{k,\varepsilon}$ satisfies

$$A_\varepsilon v \cdot \nabla_y T_\varepsilon - \frac{1}{2}\partial_{y_1}^2 T_\varepsilon - \frac{1}{2}\varepsilon^2 \partial_{y_2}^2 T_\varepsilon = \varepsilon^2\,. \tag{I.2.3.7}$$

Examining (I.2.3.7) we see that in the horizontal direction the diffusion has strength 1. However, since we impose periodic boundary conditions in this direction, there are no boundaries that provide a cooling effect directly felt by the horizontal diffusion. In the vertical direction, the diffusion coefficient is ε^2, and so the cooling effect from the Dirichlet boundary ∂S_2 will be felt in the domain in time $O(1/\varepsilon^2)$. Since our source (the right hand side of (I.2.3.7)) is also ε^2, we expect that the diffusion alone will ensure T_ε is of size $O(1)$ as $\varepsilon \to 0$. This would lead to the bound $\mathcal{E}_q^{1,p}(\mathscr{E}) \leqslant C$, which is far from optimal.

We claim that the convection term reduces this bound dramatically. Indeed, through convection one can travel an $O(1)$ distance in the vertical direction in time $1/A_\varepsilon$. Due to our no flow requirement $v \cdot \hat{n} = 0$ on ∂S_2, one can never reach the boundary of S_2 through convection alone. Thus, the cooling effect of the boundary ∂S_2 must propagate into the domain through a combination of the effects of the slow vertical diffusion $\varepsilon^2 \partial_{y_2}^2$ and the fast convection $A_\varepsilon v \cdot \nabla_y$. Our aim is to estimate how much improvement this can provide over the crude $O(1)$ bound that can be obtained through diffusion alone. This is our next result.

Proposition I.2.3.1. *There exists a smooth Hamiltonian H satisfying* (I.2.3.4a)–(I.2.3.4c)*, and a constant C such that the following holds. For every $\nu > 0$, and A_ε chosen such that $A_\varepsilon \geqslant 1/\varepsilon^\nu$ we have,*

$$\|T_\varepsilon\|_{L^\infty} \leqslant C\varepsilon^2 \Big(1 + \frac{|\ln \varepsilon|^{13}}{\varepsilon\sqrt{A_\varepsilon}}\Big) \tag{I.2.3.8}$$

for all sufficiently small ε. Here $T_\varepsilon = T^{v^\varepsilon}$, and v^ε is given by (I.2.3.5)

Remark I.2.3.2. We believe the bound (I.2.3.8) is true for every smooth, non-degenerate cellular flow v (with a constant $C = C(v)$), provided $\nu \geqslant 2$. To obtain (I.2.3.8) for all $\nu > 0$, our proof requires the velocity field v to be exactly linear near the vertical cell boundaries. We do not know whether (I.2.3.8) remains true for $\nu \in (0, 2)$ without this assumption. We note, however, that choosing $\nu \in (0,2)$ does not lead to an improved bound as in this range the constant term on the right of (I.2.3.8) will eliminate any benefit obtained from further increasing the amplitude.

REMARK I.2.3.3. For simplicity, the velocity field we construct to prove Proposition I.2.3.1 will be chosen to be exactly linear near cell corners. This assumption is mainly present as it leads to a technical simplification of the proof of Proposition I.2.3.1. Since the proof of Proposition I.2.1.3 only requires us to produce one velocity field v satisfying (I.2.3.2), we only state and prove Proposition I.2.3.1 for a specific cellular flow, instead of generic cellular flows.

We prove Proposition I.2.3.1 using probabilistic techniques in the next section. Proposition I.2.1.3 follows immediately from Proposition I.2.3.1 by scaling.

PROOF OF PROPOSITION I.2.1.3. By definition, we have

$$v^\varepsilon(x_1, x_2) = \frac{A_\varepsilon}{\varepsilon} \nabla^\perp H^\varepsilon(x_1, x_2) = \frac{A_\varepsilon}{\varepsilon^2} \begin{pmatrix} \varepsilon v_1(y_1, y_2) \\ v_2(y_1, y_2) \end{pmatrix} ,$$

and hence

$$\nabla_x v^\varepsilon = \frac{A_\varepsilon}{\varepsilon^3} \begin{pmatrix} \varepsilon \partial_{y_1} v_1 & \varepsilon^2 \partial_{y_2} v_1 \\ \partial_{y_1} v_2 & \varepsilon \partial_{y_2} v_2 \end{pmatrix}$$

Therefore, as $\varepsilon \to 0$, we have

$$\mathscr{E} = \|v^\varepsilon\|_{W^{1,p}} = O\Big(\frac{A_\varepsilon}{\varepsilon^3}\Big) .$$

Choosing $A_\varepsilon = 1/\varepsilon^\nu$, we have for large enough $\mathscr{E}$,

$$\mathscr{E} = O\Big(\frac{1}{\varepsilon^{3+\nu}}\Big) \quad \text{and} \quad \varepsilon = O\Big(\frac{1}{\mathscr{E}^{1/(3+\nu)}}\Big) \tag{I.2.3.9}$$

Combining this with (I.2.3.8), we have

$$\|T_\varepsilon\|_{L^\infty} \leqslant C\Big(\varepsilon^2 + \varepsilon^{1+\nu/2} |\ln \varepsilon|^{13}\Big) .$$

Rewriting this in terms of $\mathscr{E}$ using (I.2.3.9) and choosing $\nu = 2$ shows

$$\|T_\varepsilon\|_{L^\infty} \leqslant C \frac{|\ln \mathscr{E}|^{13}}{\mathscr{E}^{2/5}} .$$

This implies (I.2.1.7) as desired. □

I.2.4. Exit time from tall and thin cells (proof of Proposition I.2.3.1)

Our aim in this section is to prove Proposition I.2.3.1. For ease of notation we will now write $v = v^\varepsilon$, $T = T_\varepsilon$, $A = A_\varepsilon$. Let Z^ε be a solution to the SDE

$$dZ_t^\varepsilon = A v(Z^\varepsilon)\, ds + \sigma\, dB_t , \qquad \text{where} \qquad \sigma \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} . \tag{I.2.4.1}$$

Here B is a standard two dimensional Brownian motion. For convenience let $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$, and let

$$\tau^\varepsilon = \inf\{t \mid Z_{2,t}^\varepsilon \notin (-1,1)\} \tag{I.2.4.2}$$

be the first exit time of Z^ε from the strip S_2. (Here the notation $Z_{2,t}^\varepsilon$ refers to $(Z_2^\varepsilon)_t$, the value of the process Z_2^ε at time t.) By the Dynkin formula we know $T_\varepsilon(z) = \varepsilon^2 \boldsymbol{E}^z \tau^\varepsilon$.

Before delving into the details of the proof of Proposition I.2.3.1, we now briefly explain the main idea. Consider many tracer particles evolving according to (I.2.4.1). First, we note that particles near ∂S_2 get convected away from ∂S_2 in time $O(1/A)$. In this time, these particles can travel a distance of $O(\varepsilon/\sqrt{A})$ in the vertical direction through diffusion. Thus, if we can ensure particles get to within a distance of $O(\varepsilon/\sqrt{A})$ from ∂S_2, then they will exit quickly with probability at least p_0, for some small $p_0 > 0$ that is independent of ε.

We claim that in the boundary layer, every $O(1/\sqrt{A})$ seconds[1] tracer particles will pass within a distance of $O(\varepsilon/\sqrt{A})$ from ∂S. Every pass has an $O(\varepsilon)$ probability of being within $\varepsilon/\sqrt{A}$ away from ∂S_2, and so a probability $O(\varepsilon)$ of exiting from ∂S_2. This suggests

$$\sup_{z \in S_2} \boldsymbol{E}^z \tau^\varepsilon \leqslant C\Big(1 + \frac{\varepsilon}{\sqrt{A}} + \frac{(1-\varepsilon)2\varepsilon}{\sqrt{A}} + \frac{(1-\varepsilon)^2 3\varepsilon}{\sqrt{A}} + \cdots\Big) = C\Big(1 + \frac{1}{\varepsilon\sqrt{A}}\Big), \tag{I.2.4.3}$$

which is dramatically better than the crude $O(1/\varepsilon^2)$ bound obtained by using diffusion alone.

A second look at the above argument suggests that (I.2.4.3) should have a logarithmic correction. Indeed, the flow v has hyperbolic saddles at cell $\{-1,0,1\} \times \mathbb{Z}$ which causes a logarithmic slow down of particles close to it. As a result, we are able to prove the following bound on $\boldsymbol{E}\tau^\varepsilon$.

Proposition I.2.4.1. *Let $\nu > 0$ and $A \geqslant 1/\varepsilon^\nu$. There exists a cellular flow v and a constant C such that*

$$\sup_{z \in S_2} \boldsymbol{E}^z \tau^\varepsilon \leqslant C\Big(1 + \frac{|\ln \varepsilon|^{13}}{\varepsilon\sqrt{A}}\Big), \tag{I.2.4.4}$$

holds for all sufficiently small ε.

Of course Proposition I.2.4.1 immediately implies Proposition I.2.3.1.

Proof of Proposition I.2.3.1. Since $T(z) = \varepsilon^2 \boldsymbol{E}^z \tau^\varepsilon$, the estimate (I.2.4.4) implies (I.2.3.8) as desired. □

[1] The diffusion may carry particles into the interior of the cell before they exit at ∂S_2. These particles will now take $O(1/\sqrt{A})$ time to return to the boundary layer, which is why the time taken here is $O(1/\sqrt{A})$, and not the convection time $O(1/A)$.

We now describe the flow v that will be used in Proposition I.2.4.1. As remarked earlier, we expect Proposition I.2.4.1 to hold for any generic non-degenerate cellular flow. However, the specific form we describe below simplifies many technicalities. For notational convenience, we will now restrict our attention to the rectangle

$$\Omega' \stackrel{\text{def}}{=} (0,2) \times (-1,1)\,. \tag{I.2.4.5}$$

Assumption 1: The function $H\colon \mathbb{R}^2 \to [-1,1]$ is C^2 with $\|H\|_{C^2} \leqslant 100$ and is 2-periodic in both x_1 and x_2. The level set $\{H=0\}$ is precisely $(\mathbb{R}\times\mathbb{Z})\cup(\mathbb{Z}\times\mathbb{R})$. Moreover, $H(1/2,1/2)=1$, $H(3/2,1/2)=-1$ and these both correspond to non-degenerate critical points of H. All other critical points of H are hyperbolic and lie on the integer lattice $\mathbb{Z}^2$.

Assumption 2: There exists $c_0 \in (0,1/10)$ such that for

$$Q_0 \stackrel{\text{def}}{=} (-2c_0, 2c_0)^2 \tag{I.2.4.6}$$

we have

$$H(x_1,x_2) = \begin{cases} x_1 x_2 & (x_1,x_2) \in Q_0\,,\\ (1-x_1)x_2 & (x_1,x_2) \in Q_0 + (1,0)\,,\\ x_1(1-x_2) & (x_1,x_2) \in Q_0 + (0,1)\,,\\ (1-x_1)(1-x_2) & (x_1,x_2) \in Q_0 + (1,1)\,. \end{cases} \tag{I.2.4.7}$$

Assumption 3: There exists a constant h_0 such that for $x \in \{|H| < h_0\}$ and $i \in \{1,2\}$,

$$\operatorname{sign} \partial_i^2 H = -\operatorname{sign} H\,.$$

Assumption 4: In the region $\{|H| \leqslant h_0\} \cap (i + (-c,c)) \times \mathbb{R}$, where $i \in \mathbb{Z}$,

$$\partial_1 v_2 = -\partial_1^2 H = 0\,. \tag{I.2.4.8}$$

Apart from non-degeneracy and normalization, the main content of the first assumption is that H only has one critical point in the interior of every square of side length 1 with vertices on the integer lattice. This is the main geometric restriction imposed on the Hamiltonian H. Assumptions 2–3 are not necessary, but lead to technical simplifications of the proof. Finally, Assumption 4 is only required for the exit time bounds we obtain (Lemma I.2.4.2, below) to be valid when $A \leqslant 1/\varepsilon^2$. Notice that in the proof of Proposition I.2.1.3 we only use $A \approx 1/\varepsilon^2$, and so Assumption 4 is not essential. We elaborate on this in Remark I.2.4.3, below.

Now we split the proof of Proposition I.2.4.1 into two steps: estimating the time taken to reach the boundary layer, and then estimating the time taken to exit from the boundary layer. In time $1/A$, the process Z^ε will typically travel a distance of

$$\delta \stackrel{\text{def}}{=} \frac{\varepsilon}{\sqrt{A}}\,,$$

in the vertical direction. Given $\alpha > 0$ define the boundary layer $\mathcal{B}_\alpha$ by

$$\mathcal{B}_\alpha = \mathcal{B}_\alpha^\varepsilon \stackrel{\text{def}}{=} \Big\{ |H| < \frac{\alpha}{\sqrt{A}} \Big\}.$$

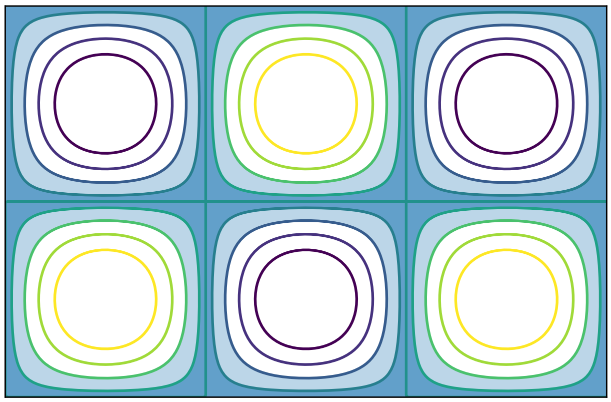

FIGURE I.2.4.1. Boundary layer $\mathcal{B}_1$ (dark blue) and boundary layer $\mathcal{B}_5$ (union of light and dark blue).

LEMMA I.2.4.2. *Let $\nu > 0$ and suppose $A \geqslant 1/\varepsilon^\nu$. There exists a constant C such that*

$$\sup_{z \in \bar{\mathcal{B}}_1} \boldsymbol{E}^z \tau^\varepsilon \leqslant \frac{C|\ln \delta|^{13}}{\varepsilon \sqrt{A}}\,. \tag{I.2.4.9}$$

Here $\bar{\mathcal{B}}_1$ denotes the closure of $\mathcal{B}_1$.

REMARK I.2.4.3. In the proof of Lemma I.2.4.2 we will see that if H doesn't satisfy Assumption 4, then Lemma I.2.4.2 is only valid if $\nu \geqslant 2$ (see Remark 2.A.5, below). It turns out that choosing $\nu \leqslant 2$ provides no additional advantage in the proof of Proposition I.2.4.1. This is because when $\nu \leqslant 2$, the constant term on the right of (I.2.4.4) dominates, and we get no improvement on $\boldsymbol{E}^z \tau^\varepsilon$.

LEMMA I.2.4.4. *For $\alpha > 0$ define*

$$\eta_\alpha = \eta_\alpha^\varepsilon \stackrel{\text{def}}{=} \inf\Big\{ t > 0 \;\Big|\; Z_t^\varepsilon \in \partial \mathcal{B}_\alpha \Big\} \tag{I.2.4.10}$$

be the first time the process Z_t^ε hits $\partial\mathcal{B}_\alpha$. There exists a constant C, independent of α, such that

$$\sup_{z \in \mathcal{B}_\alpha^c} \boldsymbol{E}^z \eta_\alpha^\varepsilon \leqslant C$$

for all sufficiently small ε. (Here $\mathcal{B}_\alpha^c$ is the complement of $\mathcal{B}_\alpha$.)

A proof of Lemma I.2.4.4 using a blow-up argument can be found in [IS12]. We present a different proof of this fact (in Section I.2.6, below) by constructing a supersolution based on the Freidlin averaging problem [FW12].

Momentarily postponing the proofs of Lemmas I.2.4.2 and I.2.4.4, we prove Proposition I.2.4.1.

PROOF OF PROPOSITION I.2.4.1. If $z \notin \mathcal{B}_1$, the strong Markov property, Lemmas I.2.4.2 and I.2.4.4 imply

$$\begin{aligned} \boldsymbol{E}^z \tau^\varepsilon &= \boldsymbol{E}^z \eta_1^\varepsilon + (\tau^\varepsilon - \eta_1^\varepsilon) = \boldsymbol{E}^z \Big(\eta_1^\varepsilon + (\tau^\varepsilon - \eta_1^\varepsilon) \mid \mathcal{F}_{\eta_1^\varepsilon} \Big) \\ &\leqslant C + \boldsymbol{E}^z \sup_{z' \in \bar{B}_1} \boldsymbol{E}^{z'} \tau^\varepsilon \leqslant C \Big(1 + \frac{|\ln \delta|^{13}}{\varepsilon \sqrt{A}} \Big) . \end{aligned} \tag{I.2.4.11}$$

If $z \in \mathcal{B}_1$, then Lemma I.2.4.2 directly implies (I.2.4.11). Thus in either case we have (I.2.4.4), as desired. □

I.2.5. Exit from the Boundary layer

In this section, we will prove Lemma I.2.4.2. We will fix $\nu > 0$ and suppose $A \geqslant 1/\varepsilon^\nu$ as in the hypothesis of Lemma I.2.4.2 through out this section. Furthermore, for notational convenience, we will now drop the explicit ε dependence from Z^ε and A.

The main idea behind the proof of Lemma I.2.4.2 is to focus our attention on trajectories in the boundary layer $\mathcal{B}_1$, until they leave the bigger boundary layer $\mathcal{B}_5$. Our first lemma estimates the chance of starting in $\mathcal{B}_1$ and exiting the strip S_2, before exiting the bigger boundary layer $\mathcal{B}_5$.

LEMMA I.2.5.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\inf_{z \in \mathcal{B}_1} \boldsymbol{P}^z (\tau^\varepsilon < \eta_5^\varepsilon) \geqslant \frac{C\varepsilon}{|\ln \delta|^{12}} \tag{I.2.5.1}$$

for all sufficiently small ε.

Our next lemma estimates the amount of time the process spends in the bigger boundary layer $\mathcal{B}_5$ (light blue region in Figure I.2.4.1).

LEMMA I.2.5.2. *There exists a constant C such that*

$$\sup_{z \in \mathcal{B}_1} \boldsymbol{E}^z \eta_5^\varepsilon \leqslant \frac{C |\ln \delta|}{A} \tag{I.2.5.2}$$

for all sufficiently small ε.

Finally, we estimate the time taken for the process to return to the boundary layer $\mathcal{B}_1$ starting from the boundary of the bigger boundary layer $\mathcal{B}_5$. This is the slowest step, taking time $O(|\ln \delta| / \sqrt{A})$ instead of $O(|\ln \delta| / A)$.

LEMMA I.2.5.3. *There exists a constant C such that there exists an ε_0, where*

$$\sup_{z\in\partial\mathcal{B}_5} \boldsymbol{E}^z \eta_1^\varepsilon \leqslant C\frac{|\ln\delta|}{\sqrt{A}} \tag{I.2.5.3}$$

for all $\varepsilon < \varepsilon_0$.

Momentarily postponing the proofs of Lemmas I.2.5.1–I.2.5.3, we prove Lemma I.2.4.2.

PROOF OF LEMMA I.2.4.2. In this proof, the constant C may vary from line to line but does not depend on ε. We first define two sequences of barrier stopping times,

$$\sigma_0' = 0\,, \qquad\qquad \tilde\sigma_0 = \inf\Big\{t \geqslant \sigma_0' \,\Big|\, Z_t^\varepsilon \in \partial\mathcal{B}_5\Big\}\,,$$
$$\sigma_n' = \inf\Big\{t \geqslant \tilde\sigma_{n-1} \,\Big|\, Z_t^\varepsilon \in \partial\mathcal{B}_1\Big\}\,, \qquad \tilde\sigma_n = \inf\Big\{t \geqslant \sigma_n' \,\Big|\, Z_t^\varepsilon \in \partial\mathcal{B}_5\Big\}\,.$$

We have

$$\begin{aligned}
\boldsymbol{E}^z\tau^\varepsilon &= \int_0^\infty \boldsymbol{P}^z\Big(\tau^\varepsilon \geqslant t\Big)\,dt\\
&= \boldsymbol{E}^z \sum_{n=1}^\infty \int_{\sigma_{n-1}'}^{\sigma_n'} \boldsymbol{1}_{\{\tau^\varepsilon\geqslant t\}}\,dt \leqslant \sum_{n=1}^\infty \boldsymbol{E}^z \boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_{n-1}'\}}(\sigma_n' - \sigma_{n-1}')\\
&= \sum_{n=1}^\infty \boldsymbol{E}^z \boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_{n-1}'\}} \boldsymbol{E}^{Z^\varepsilon(\sigma_{n-1}')}\sigma_1'\\
&\leqslant \sum_{n=1}^\infty \boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_{n-1}') \sup_{z'\in\partial\mathcal{B}_1} \boldsymbol{E}^{z'}\sigma_1'\,.
\end{aligned} \tag{I.2.5.4}$$

We will now estimate each term on the right.

First, by the strong Markov property and Lemmas I.2.5.2–I.2.5.3 we have

$$\boldsymbol{E}^z\sigma_1' = \boldsymbol{E}^z\Big(\tilde\sigma_0 + \boldsymbol{E}^{Z^\varepsilon(\tilde\sigma_0)}\eta_1^\varepsilon\Big) \leqslant \boldsymbol{E}^z\Big(\eta_5^\varepsilon + \sup_{z'\in\partial\mathcal{B}_5}\boldsymbol{E}^{z'}\eta_1^\varepsilon\Big) \leqslant \frac{C|\ln\delta|}{\sqrt{A}}\,. \tag{I.2.5.5}$$

for every $z \in \partial\mathcal{B}_1$. To estimate $\boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_n')$, we use Lemma I.2.5.1 and the fact that $\sigma_1' \geqslant \tilde\sigma_0 = \eta_5^\varepsilon$ to obtain

$$\sup_{z\in\partial\mathcal{B}_1}\boldsymbol{P}^z(\tau^\varepsilon \geqslant \sigma_1') \leqslant \sup_{z\in\partial\mathcal{B}_1}\boldsymbol{P}^z(\tau^\varepsilon \geqslant \eta_5^\varepsilon) = 1 - \inf_{z\in\partial\mathcal{B}_1}\boldsymbol{P}^z(\tau^\varepsilon < \eta_5^\varepsilon) \leqslant 1 - \frac{C\varepsilon}{(\ln\delta)^{12}}\,.$$

Now, by the strong Markov property,

$$\begin{aligned}
\sup_{z\in\mathcal{B}_1}\boldsymbol{P}^z\Big(\tau^\varepsilon \geqslant \sigma_n'\Big) &= \sup_{z\in\mathcal{B}_1}\boldsymbol{E}^z\Big(\boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_{n-1}'\}}\boldsymbol{E}^{Z^\varepsilon(\sigma_{n-1}')}\boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_1'\}}\Big)\\
&\leqslant \sup_{z\in\mathcal{B}_1}\boldsymbol{E}^z\boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_{n-1}'\}} \sup_{z'\in\partial\mathcal{B}_1}\boldsymbol{P}^{z'}(\tau^\varepsilon \geqslant \sigma_1')\\
&\leqslant \Big(1 - \frac{C\varepsilon}{(\ln\delta)^{12}}\Big)\boldsymbol{E}^z\boldsymbol{1}_{\{\tau^\varepsilon\geqslant\sigma_{n-1}'\}}\,.
\end{aligned}$$

Hence by induction

$$\sup_{z\in\mathcal{B}_1} \boldsymbol{P}^z\Big(\tau^\varepsilon \geqslant \sigma'_n\Big) \leqslant \Big(1 - \frac{C\varepsilon}{|\ln\delta|^{12}}\Big)^n, \tag{I.2.5.6}$$

for all $n \in \mathbb{N}$.

Using (I.2.5.5) and (I.2.5.6) in (I.2.5.4) yields

$$\boldsymbol{E}^z\tau^\varepsilon \leqslant \frac{C|\ln\delta|}{\sqrt{A}} \sum_{n=0}^{\infty}\Big(1 - \frac{C\varepsilon}{|\ln\delta|^{12}}\Big)^n$$

finishing the proof. □

I.2.5.1. Proof of Lemma I.2.5.1. In this subsection, we will give the proof of Lemma I.2.5.1. Let the coordinate processes of Z be Z_1 and Z_2 respectively (i.e. $Z = (Z_1, Z_2)$). Define γ_t to be the deterministic curve satisfying the ODE

$$\partial_t\gamma_t = Av(\gamma_t)\,. \tag{I.2.5.7}$$

We again need a few results to prove Lemma I.2.5.1.

By symmetry and the reflection principle, when Z wanders into the lower half of the domain $(0, 2) \times (-1, 0)$, its behavior is mirrored by $-Z$, which is again on the upper half of the domain $(0, 2) \times (0, 1)$. Hence, without loss of generality, we may restrict our attention to the upper half of the domain and all the lemmas below are stated in this context.

The first result we state is a “tube lemma” estimating the probability that the process Z stays within a small tube around the deterministic trajectories. This is well studied and many such estimates can be found in the literature (see for instance [FW12]). The standard estimates, however, work well for times of order $1/A$. Due to the degeneracy, and the hyperbolic saddles near cell corners, we need an estimate that works for time scales of order $|\ln\delta|/A$. We state this estimate here.

Lemma I.2.5.4. *Let $z_0 \in (0, 2)\times(0, 1)\cap\Big(Q_0/2+(j, k)\Big)$ where $(j, k) \in \{0, 1, 2\}\times\{0, 1\}$ and Q_0 is as in (I.2.4.6). Let γ satisfy (I.2.5.7) with $\gamma_0 = z_0$, and define*

$$T \stackrel{\text{def}}{=} \inf\{t > 0 \mid |\gamma_{2,t} - 1| \leqslant \delta \text{ or } |\gamma_{1,t} - 1| = c_0 \text{ or } |\gamma_{2,t} - 1| = c_0\}\,. \tag{I.2.5.8}$$

Then there exists ε_0 so that for every $\varepsilon < \varepsilon_0$,

$$P^{z_0}\Big(\sup_{0\leqslant t\leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_{ii}}{\sqrt{|\ln\delta|A}}\,, \forall i \in \{1, 2\}\Big) \geqslant \frac{C}{|\ln\delta|^2}\,.$$

Here we recall that $\sigma_{11} = 1$ and $\sigma_{22} = \varepsilon$ are the diagonal entries in the matrix σ in (I.2.4.1).

Remark I.2.5.5. By a direct calculation, we can check that $T \leqslant |\ln\delta|/A$.

The proof of Lemma I.2.5.4 uses the Girsanov theorem and is greatly simplified by the fact that H is exactly quadratic near cell corners. Since it is similar to the standard proofs, we present it in Appendix 2.A.

Once Lemma I.2.5.4 is established it quickly gives an estimate on the probability of getting within a distance of $O(1/\sqrt{A})$ away from cell boundaries.

LEMMA I.2.5.6. *Let $z_0 \in \mathcal{B}_1 \cap (0,2) \times (0,1)$. There exist constants $C, M > 0$ such that for small enough ε,*

$$\boldsymbol{P}^{z_0}\Big(\lambda_0 < \eta^{\varepsilon}_{4M}\Big) \geqslant \frac{C}{|\ln \delta|^2}\,. \tag{I.2.5.9}$$

Here, $\lambda_0 \stackrel{\text{def}}{=} \inf\Big\{t > 0 \;\Big|\; Z_t \in \{\operatorname{dist}(z, \partial\Omega') \leqslant M/\sqrt{A}\}\Big\}$.

PROOF. Note first that by Taylor expansion of H, for small ε there exists $M > 0$ such that $\operatorname{dist}(z_0, \partial\Omega') \leqslant M/\sqrt{A}$ for all z_0 outside the corners $Q_0/2 + (j,k)$, where $(j,k) \in \{0,1,2\} \times \{0,1\}$. So now, we assume $z_0 \in Q_0/2 + (j,k)$ for some $(j,k) \in \{0,1,2\} \times \{0,1\}$. For brevity, we only present the proof when $z_0 \in Q_0/2$, as the other cases are identical.

If $\operatorname{dist}(z_0, \partial\Omega') \leqslant 1/\sqrt{A}$ we are done, so we now suppose $z_0 \in Q_0/2$ with $\operatorname{dist}(z_0, \partial\Omega') > 1/\sqrt{A}$. Let γ be the deterministic trajectory defined by (I.2.5.7) with $\gamma_0 = z_0$, and let T be as in (I.2.5.8). Note that since $\operatorname{dist}(z_0, \partial\Omega') > 1/\sqrt{A}$ we can not have $|\gamma_{2,T} - 1| \leqslant \delta$. Thus, either $|\gamma_{1,T} - 1| = c_0$ or $|\gamma_{2,T} - 1| = c_0$. In either case there exists a constant M such that $|\gamma_{2,T} - 1| \leqslant M/\sqrt{A}$ or $|\gamma_{1,T} - 1| \leqslant M/\sqrt{A}$, respectively. Now using Lemma I.2.5.4 we obtain (I.2.5.9) as desired. ☐

REMARK I.2.5.7. For notational convenience, we assume that $M = 1$ for the rest of the chapter.

Another consequence of Lemma I.2.5.4 is a lower bound on the probability of reaching $O(\delta)$ away from the top boundary before re-entering the cell interior.

LEMMA I.2.5.8. *Let $Q^{\delta}_{top} = (1 - 2c_0, 1 + 2c_0) \times (1 - 4\delta, 1)$ be a box of height 4δ at the top of the cell corner. Let $\lambda \stackrel{\text{def}}{=} \inf\{t \geqslant 0 \mid Z_t \in Q^{\delta}_{top}\}$. Then, there exists a constant $C > 0$ such that*

$$\inf_{z_0 \in (1-\delta, 1+\delta) \times (1 - c_0, 1)} \boldsymbol{P}^{z_0}\Big(\lambda < \eta^{\varepsilon}_4\Big) \geqslant \frac{C}{(\ln \delta)^2}\,. \tag{I.2.5.10}$$

PROOF. Let $T = \inf\Big\{t > 0 \;\Big|\; |\gamma_{2,t} - 1| \leqslant \delta\Big\}$ the time the deterministic process hits the top boundary layer with width δ. By Lemma I.2.5.4, there exists a constant $C > 0$ so that

$$P^{z_0}\Big(\sup_{0 \leqslant t \leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_{ii}}{\sqrt{|\ln \delta| A}}\,, \forall i \in \{1,2\}\Big) \geqslant \frac{C}{(\ln \delta)^2}\,.$$

As $z_0 \in (1-\delta, 1+\delta) \times (1-c_0, 1)$, $\gamma_{1,T} \in (1-c_0, 1+c_0)$. Therefore,

$$\Big\{ \sup_{0 \leqslant t \leqslant T} |Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_{ii}}{\sqrt{|\ln \delta| A}}, \forall i \in \{1,2\} \Big\} \subseteq \Big\{ \eta_4^\varepsilon > \lambda \Big\},$$

from which (I.2.5.10) follows. ☐

Next, we bound the probability of exiting from the top when trajectories start in Q^δ_{top}.

LEMMA I.2.5.9. *There exists a constant $p_0 > 0$ such that*

$$\inf_{z_0 \in Q^\delta_{top}} \boldsymbol{P}^{z_0}\Big(\tau^\varepsilon < \eta_4^\varepsilon\Big) \geqslant p_0 \,. \tag{I.2.5.11}$$

PROOF. Let $\tilde{T} = 1/A$. When A is sufficiently large, we note that given $X_0 = z_0 \in Q^\delta_{\text{top}}$, there exists $n \geqslant 1$, independent of ε, such that the deterministic flow γ_t starting at z_0 still remains in the top edge of the boundary layer $\{|H| \leqslant n\delta\} \cap (0,2) \times (1-n\delta, 1)$ for time $\tilde{T}$. Define $\tilde{\gamma}_t$ by

$$\partial_t \tilde{\gamma}_t = A u(\tilde{\gamma}_t)\,,$$

where $u = (u_1, u_2)$ is chosen to satisfy the following condition $\tilde{\gamma}_t = (\gamma_{1,t}, \tilde{\gamma}_{2,t})$, where $\gamma_{1,t}$ is the first coordinate of γ, and $\tilde{\gamma}_{2,t}$ is some continuous function such that

$$\tilde{\gamma}_{2,0} = \gamma_{2,0}\,, \qquad |v_2 - u_2| \leqslant 2n\delta \qquad \text{and} \qquad \tilde{\gamma}_{2,\tilde{T}} \geqslant n\delta\,.$$

An example of such $\tilde{\gamma}$ is $\tilde{\gamma}_t = (\gamma_{1,t}, \gamma_{2,t} + 2An\delta t)$. By continuity of Z, we have

$$E_3 \stackrel{\text{def}}{=} \Big\{ \sup_{0 \leqslant t \leqslant \tilde{T}} |Z_{2,t} - \tilde{\gamma}_{2,t}| \leqslant \delta \Big\} \subset \Big\{ \tau^\varepsilon < \eta_4^\varepsilon \Big\}\,.$$

Now a standard large deviation estimate will show that $\boldsymbol{P}^{z_0}(E_3) \geqslant p_\varepsilon$, for some constant C_ε that vanishes as $\varepsilon \to 0$. In order to prove Lemma I.2.5.9, we need to remove this ε dependence. We do this here using the fact that in this box $|\partial_1 v_2| \leqslant O(\varepsilon)$, and $|v_2 - u_2| \leqslant O(\delta)$. We claim that if we go through the standard large deviation estimate with these additional assumptions, the constant p_ε can be made independent of ε. Since the details are not too different from the standard proof, we carry them out in Lemma 2.A.3 in Appendix 2.A, below. Hence, we see that there exists a constant p_0 (independent of z_0, ε) so that

$$\boldsymbol{P}^{z_0}(E_3) \geqslant p_0\,,$$

proving (I.2.5.11). ☐

LEMMA I.2.5.10. *Let $\tilde{\lambda} \stackrel{\text{def}}{=} \inf\Big\{ t \geqslant 0 \,\Big|\, Z_t \in (1-\delta, 1+\delta) \times (1-c_0, 1) \Big\}$. There exists a constant $C > 0$ such that*

$$\inf_{z_0 \in \{\text{dist}(z, \partial\Omega') \leqslant 1/\sqrt{A}\}} \boldsymbol{P}^{z_0}\Big(\tilde{\lambda} < \eta_4^\varepsilon\Big) \geqslant \frac{C\varepsilon}{(\ln \delta)^8}\,. \tag{I.2.5.12}$$

PROOF. We give the proof where $z_0 \in \{\operatorname{dist}(z, \partial\Omega') \leqslant 1/\sqrt{A}\} \cap (0,1) \times (0,1)$. The analysis is similar for $z_0 \in \{\operatorname{dist}(z, \partial\Omega') \leqslant 1/\sqrt{A}\} \cap (1,2) \times (0,1)$. Define the regions $\square_1, \ldots, \square_5$ by

$$\square_1 \stackrel{\text{def}}{=} \left(1 - \frac{1}{\sqrt{A}}, 1 + \frac{1}{\sqrt{A}}\right) \times \left(\frac{1}{\sqrt{A}}, 1 - \frac{1}{\sqrt{A}}\right),$$

$$\square_2 \stackrel{\text{def}}{=} \left(\frac{1}{\sqrt{A}}, 1\right) \times \left(0, \frac{1}{\sqrt{A}}\right),$$

$$\square_3 \stackrel{\text{def}}{=} \left(0, \frac{1}{\sqrt{A}}\right) \times \left(0, 1 - \frac{1}{\sqrt{A}}\right),$$

$$\square_4 \stackrel{\text{def}}{=} \left(0, 1 - \frac{1}{\sqrt{A}}\right) \times \left(1 - \frac{1}{\sqrt{A}}, 1\right),$$

$$\square_5 \stackrel{\text{def}}{=} \left(1 - \frac{1}{\sqrt{A}}, 0\right)^2,$$

as shown in Figure I.2.5.1. If $\operatorname{dist}(z_0, \partial\Omega') \leqslant 1/\sqrt{A}$, then z_0 must be in one

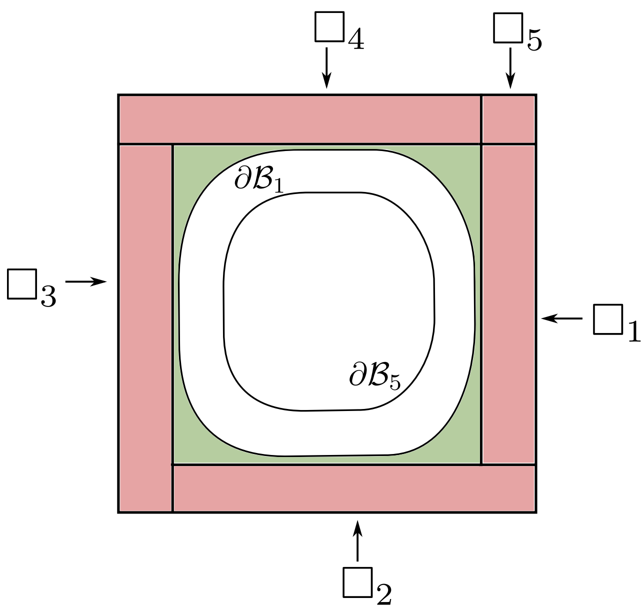

FIGURE I.2.5.1. $\partial\mathcal{B}_n$ and $\square_i$.

of the boxes $\square_1, \ldots, \square_5$. Suppose first $z_0 \in \square_1$. Let $\gamma(t)$ is the deterministic trajectory such that $\gamma_0 = z_0$, $T_0 \stackrel{\text{def}}{=} \inf\left\{t > 0 : \gamma_{2,t} = 1 - c_0/2\right\} \leqslant m/A$ for some $m \geqslant 1$, and

$$E_4 \stackrel{\text{def}}{=} \left\{ \sup_{0 \leqslant t \leqslant T_0} |Z_{1,t} - \gamma_{1,t}| \leqslant \frac{2}{\sqrt{A}}\,, \sup_{0 \leqslant t \leqslant T_0} |Z_{2,t} - \gamma_{2,t}| \leqslant \frac{\varepsilon}{\sqrt{A}}\,, |Z_{1,T_0}| \leqslant \frac{\varepsilon}{2\sqrt{A}}\,, \right\}.$$

By continuity, we have that

$$E_4 \subset \left\{\tilde{\lambda} < \eta_4^{\varepsilon}\right\}.$$

We claim

$$\boldsymbol{P}^{z_0}\Big(\tilde{\lambda} < \eta_4^\varepsilon\Big) \geqslant \boldsymbol{P}^{z_0}(E_4) \geqslant C\varepsilon\,, \tag{I.2.5.13}$$

where $C > 0$ independent of z_0. The proof of (I.2.5.13) is presented with the other tube lemmas we use in Appendix 2.A. We in fact prove a more general estimate (Lemma 2.A.4 applied to the deterministic flow), from which (I.2.5.13) follows.

Now, let $z_0 \in \square_2$, define $\square_{2R} = \square_2 \cap [1-c_0, 1] \times [0, 2/\sqrt{A}]$, and let $\lambda_1 = \inf\Big\{t > 0 \,\Big|\, Z_t \in \square_{2R}\Big\}$. Proceeding as the case for $\square_1$ with $\gamma(t)$ being the deterministic trajectory so that $\gamma(0) = z_0$, $T_1 = \inf\{t > 0 \mid \gamma_{1,t} = c_0/2\}$, we have

$$\boldsymbol{P}^{z_0}\Big(\lambda_1 < \eta_4^\varepsilon\Big) \geqslant \boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant T_1} |Z_t - \gamma_t| \leqslant \frac{1}{\sqrt{A}}\Big) \geqslant C\,. \tag{I.2.5.14}$$

To see why the last lower bound is true, we consider by Itô formula,

$$\sup_{0\leqslant t\leqslant T_1} \boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant 2A\|v\|_{C^1} \int_0^{T_1} \boldsymbol{E}^{z_0} \sup_{0\leqslant t\leqslant T_1} |Z_t - \gamma_t|^2 + (\varepsilon^2+1)T_1,$$

which, by Gronwall's inequality and Assumption 1, implies

$$\sup_{0\leqslant t\leqslant T_1} \boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant (1+\varepsilon^2)T_1 e^{200T_1}\,.$$

Inequality (I.2.5.14) follows by Chebychev's inequality.

Now let $\lambda' = \inf\Big\{t \geqslant 0 \,\Big|\, Z_t \in \square_1\Big\}$. Using Lemmas I.2.5.4 and Markov property, there exists a constant C (independent of z_0) so that

$$\boldsymbol{P}^{z_0}\Big(\lambda' < \eta_4^\varepsilon\Big) \geqslant \boldsymbol{P}^{z_0}\Big(\lambda_1 < \eta_4^\varepsilon\Big) \inf_{z_1\in\square_{2R}} \boldsymbol{P}^{z_1}\Big(\lambda' < \eta_4^\varepsilon\Big) \geqslant \frac{C}{(\ln\delta)^2}\,. \tag{I.2.5.15}$$

Combining (I.2.5.13), (I.2.5.15) and using the Markov property gives

$$\boldsymbol{P}^{z_0}\Big(\tilde{\lambda} < \eta_4^\varepsilon\Big) \geqslant \boldsymbol{P}^{z_0}\Big(\lambda' < \eta_4^\varepsilon\Big) \inf_{z_1\in\square_1} \boldsymbol{P}^{z_1}\Big(\tilde{\lambda} < \eta_4^\varepsilon\Big) \geqslant \frac{C\varepsilon}{(\ln\delta)^2}\,.$$

Repeating this argument again for $\square_3, \ldots, \square_5$ we see that we obtain an extra $C/|\ln\delta|^2$ factor every time we pass a corner. Combining these estimates gives (I.2.5.12) as claimed. ☐

We are now ready to give the proof for Lemma I.2.5.1.

Proof of Lemma I.2.5.1. Let $z_0 \in \mathcal{B}_1$ and denote $D_1 \stackrel{\text{def}}{=} \Big\{\operatorname{dist}(z, \partial\Omega') \leqslant 1/\sqrt{A}\Big\}$, $D_2 \stackrel{\text{def}}{=} (1-\delta, 1+\delta) \times (1-c_0, 1)$ and $D_3 \stackrel{\text{def}}{=} (1-2c_0, 1+2c_0) \times (1-4\delta, 1)$. As $\eta_4^\varepsilon < \eta_5^\varepsilon$ when $z_0 \in \mathcal{B}_1$, by Lemmas I.2.5.6–I.2.5.10 and Markov property, we have that

$$\boldsymbol{P}^{z_0}(\tau^\varepsilon < \eta_5^\varepsilon) \geqslant \boldsymbol{E}^{z_0}\mathbf{1}_{\{\tau^\varepsilon<\eta_5^\varepsilon\}}\mathbf{1}_{\{\lambda<\eta_5^\varepsilon\}}\mathbf{1}_{\{\lambda_0<\eta_5^\varepsilon\}}\mathbf{1}_{\{\tilde{\lambda}<\eta_5^\varepsilon\}}$$

$$
\begin{aligned}
&= \boldsymbol{E}^{z_0}\mathbf{1}_{\{\lambda_0<\eta_5^\varepsilon\}}\boldsymbol{E}^{z_0}\Big(\mathbf{1}_{\{\tau^\varepsilon<\eta_5^\varepsilon\}}\mathbf{1}_{\{\tilde\lambda<\eta_5^\varepsilon\}}\mathbf{1}_{\{\lambda<\eta_5^\varepsilon\}}\;\Big|\;\mathcal{F}_{\lambda_0}\Big)\\
&= \boldsymbol{E}^{z_0}\mathbf{1}_{\{\lambda_0<\eta_5^\varepsilon\}}\boldsymbol{E}^{Z_{\lambda_0}}\Big(\mathbf{1}_{\{\tau^\varepsilon<\eta_5^\varepsilon\}}\mathbf{1}_{\{\tilde\lambda<\eta_5^\varepsilon\}}\mathbf{1}_{\{\lambda<\eta_5^\varepsilon\}}\Big)\\
&\geqslant \boldsymbol{E}^{z_0}\mathbf{1}_{\{\lambda_0<\eta_5^\varepsilon\}}\inf_{z_1\in D_1}\boldsymbol{E}^{z_1}\Big(\mathbf{1}_{\{\lambda<\eta_5^\varepsilon\}}\mathbf{1}_{\{\tilde\lambda<\eta_5^\varepsilon\}}\mathbf{1}_{\{\tau^\varepsilon<\eta_5^\varepsilon\}}\Big)\\
&\geqslant \boldsymbol{E}^{z_0}\mathbf{1}_{\{\lambda_0<\eta_5^\varepsilon\}}\inf_{z_1\in D_1}\boldsymbol{E}^{z_1}\mathbf{1}_{\{\tilde\lambda<\eta_5^\varepsilon\}}\inf_{z_2\in D_2}\boldsymbol{E}^{z_2}\mathbf{1}_{\{\lambda<\eta_5^\varepsilon\}}\inf_{z_3\in D_3}\boldsymbol{E}^{z_3}\mathbf{1}_{\{\tau^\varepsilon<\eta_5^\varepsilon\}}\\
&\geqslant \frac{C\varepsilon}{|\ln\delta|^{12}}\,,
\end{aligned}
$$

where C is independent of z_0. Taking the infimum over z_0, we achieve the desired result. $\square$

I.2.5.2. Proof of Lemma I.2.5.2. In this subsection, we give a proof of Lemma I.2.5.2. The strategy then will be similar to that of the proof of Lemma I.2.5.1 as will will estimate the probability for a typical particle to successfully enter the inner region after each time it goes around the boundary layer $\mathcal{B}_5$. To do this, we first need a few results.

Lemma I.2.5.11. *Let* $\tilde\square_1 = \mathcal{B}_5\cap\{x_2\in[c_0,1-c_0]\}$. *There exists a constant* p_0 *such that*

$$
\inf_{z_0\in\tilde\square_1}\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon<\frac{1}{A}\Big)\geqslant p_0\,. \tag{I.2.5.16}
$$

Proof. Since we restrict our attention to region of the boundary layer on the sides, for each $\varepsilon>0$ there exists an interval R_ε with length $|R_\varepsilon| = 1/\sqrt{A}$ such that

$$
\operatorname{dist}\Big(R_\varepsilon\times[c_0,1-c_0]\,,\mathcal{B}_5\cap\{x_2\in[c_0,1-c_0]\}\Big)=\frac{1}{\sqrt{A}}\,.
$$

Let M be independent of ε such that

$$
R_\varepsilon\times[c_0,1-c_0]\cup\Big(\mathcal{B}_5\cap\{x_2\in[c_0,1-c_0]\}\Big)\subseteq\Big(1-\frac{M}{\sqrt{A}}\,,1+\frac{M}{\sqrt{A}}\Big)\times[c_0,1-c_0]\,,
$$

and $z_0\in\tilde\square_1$. By Lemma 2.A.4 applied to the deterministic curve γ (given by (I.2.5.7)) with $\gamma_0 = z_0$, we have

$$
\begin{aligned}
&\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon<\frac{1}{A}\Big)\\
&\geqslant \boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant 1/A}|Z_{1,t}-\gamma_{1,t}|\leqslant\frac{M}{\sqrt{A}}\,,\sup_{0\leqslant t\leqslant 1/A}|Z_{2,t}-\gamma_{2,t}|\leqslant\frac{\varepsilon}{\sqrt{A}}\,,Z_{1,T_0}\in R_\varepsilon\Big)\geqslant p_0\,,
\end{aligned}
$$

where p_0 is independent of z_0 as desired. $\square$

Lemma I.2.5.12. *Let* $\tilde{\lambda}_2 = \inf\Big\{t > 0 \;\Big|\; Z_{2,t} \in \{c_0, 1 - c_0\}\Big\}$ *and* $z_0 \in \mathcal{B}_5 - \tilde{\square}_1$. *Then*

$$\lim_{\varepsilon\to 0} \inf_{\mathcal{B}_5 - \tilde{\square}_1} \boldsymbol{P}^{z_0}\Big(\tilde{\lambda}_2 \leqslant \frac{5|\ln\delta|}{A}\Big) \geqslant 1 - \frac{C\ln A}{A^{1/4}}\,. \tag{I.2.5.17}$$

Proof. Let $q \geqslant 2$ be some large number to be chosen later, and let $\tilde{z}_0$ be the closest point on $\{H = A^{-1/q}\}$ to z_0. Let $\tilde{d} = A|z_0 - \tilde{z}_0|$ and γ_t be the deterministic curve (defined by (I.2.5.7)) with $\gamma_0 = \tilde{z}_0$. Note that, by Assumptions 1–2,

$$\frac{\tilde{d}}{A} \leqslant \frac{C}{A^{1/2q}}\,. \tag{I.2.5.18}$$

By Itô formula, we have

$$\boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant \frac{\tilde{d}^2}{A^2} + 2A\|v\|_{C^1}\int_0^t \boldsymbol{E}^{z_0}|Z_s - \gamma_s|^2\,ds + (1+\varepsilon^2)t\,.$$

By Gronwall's inequality and Assumption 1, it follows that

$$\boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant \Big(\frac{\tilde{d}^2}{A^2} + (1+\varepsilon^2)t\Big)e^{200At}\,.$$

Now, let $T_0 = \inf\{t > 0 : \gamma_{2,t} \in (2c_0, 1-2c_0)\}$, and note that $T_0 \leqslant D\ln A/(Aq)$ for some constant $D > 0$. By (I.2.5.18), we have

$$\begin{aligned}
&\boldsymbol{P}^{z_0}\Big(|Z_{T_0} - \gamma_{T_0}| \geqslant \frac{c_0}{10}\Big) \leqslant \frac{100}{c_0^2}\Big(\frac{C}{A^{2q}} + (1+\varepsilon^2)\frac{D\ln A}{Aq}\Big)e^{200D\ln A/q}\\
&\leqslant CA^{200D/q-1}\ln A\,.
\end{aligned}$$

Picking q such that $200D/q - 1 < -1/2$, we have

$$\boldsymbol{P}^{z_0}\Big(|Z_{T_0} - \gamma_{T_0}| < \frac{c_0}{10}\Big) \geqslant 1 - \frac{C\ln A}{A^{1/4}}\,. \tag{I.2.5.19}$$

As $q \geqslant 2$, $T_0 < 5|\ln\delta|/A$. Therefore, by continuity of Z, it follows that

$$\Big\{Z_{2,T_0} \in [2c_0, 1-2c_0]\Big\} \subseteq \Big\{\tilde{\lambda}_2 \leqslant \frac{5|\ln\delta|}{A}\Big\}\,.$$

Combining this with (I.2.5.19), we deduce

$$\lim_{\varepsilon\to 0} \inf_{\mathcal{B}_5 - \tilde{\square}_1} \boldsymbol{P}^{z_0}\Big(\tilde{\lambda}_2 \leqslant \frac{5|\ln\delta|}{A}\Big) \geqslant 1 - \frac{C\ln A}{A^{1/4}}\,,$$

as desired. □

We are now ready for the proof of Lemma I.2.5.2.

Proof of Lemma I.2.5.2. *Step 1:* We first claim that for each $z_0 \in \mathcal{B}_5$ and $\varepsilon > 0$, there exists a constant $C > 0$, independent of z_0 and ε, such that

$$\boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)| > \frac{5}{\sqrt{A}}\Big) \geqslant C\,. \tag{I.2.5.20}$$

To prove this, suppose for contradiction there exists a sequence $\{z_n, \varepsilon_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)| > \frac{5}{\sqrt{A}}\Big) = 0\,. \tag{I.2.5.21}$$

Let C_0 be the lower bound in Lemma I.2.5.11 and denote $\tilde{\lambda}_1 = \inf\Big\{t \geqslant 0 \;\Big|\; Z_t \in \tilde{\Box}_1\Big\}$. By Lemma I.2.5.11 and the strong Markov property,

$$\begin{aligned}
&\boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)| > \frac{5}{\sqrt{A}}\Big)\\
&\geqslant \boldsymbol{E}^{z_n}\Big(\boldsymbol{E}^{z_n}\Big(\boldsymbol{1}_{\Big\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\Big\}}\boldsymbol{1}_{\Big\{\tilde{\lambda}_1\leqslant 5|\ln\delta|/A\Big\}}\boldsymbol{1}_{\Big\{\eta_5^{\varepsilon}<\tilde{\lambda}_1+1/A\Big\}} \mid \mathcal{F}_{\tilde{\lambda}_1}\Big)\Big)\\
&= \boldsymbol{E}^{z_n}\Big(\boldsymbol{1}_{\Big\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\Big\}}\boldsymbol{1}_{\Big\{\tilde{\lambda}_1\leqslant 5|\ln\delta|/A\Big\}}\boldsymbol{E}^{Z_{\tilde{\lambda}_1}}\boldsymbol{1}_{\Big\{\eta_5^{\varepsilon}<1/A\Big\}}\Big)\\
&\geqslant \boldsymbol{E}^{z_n}\Big(\boldsymbol{1}_{\Big\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\Big\}}\boldsymbol{1}_{\Big\{\tilde{\lambda}_1\leqslant 5|\ln\delta|/A\Big\}}\Big)\inf_{z\in\tilde{\Box}_1}\boldsymbol{E}^{z}\boldsymbol{1}_{\Big\{\eta_5^{\varepsilon}<1/A\Big\}}\\
&\geqslant C_0\boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\,;\tilde{\lambda}_1 \leqslant \frac{5|\ln\delta|}{A}\Big)\,.
\end{aligned}$$

The second equality follows from the fact that $\eta_5^{\varepsilon} > \tilde{\lambda}_1$ under the event

$$\Big\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\Big\}\,.$$

We claim that for large enough n, we have

$$\boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)| \leqslant \frac{5}{\sqrt{A}}\,;\tilde{\lambda}_1 \leqslant \frac{5|\ln\delta|}{A}\Big) \geqslant \frac{1}{2}\,,$$

which contradicts our assumption (I.2.5.21). To see that this lower bound is true, we first note that $z_i \notin \tilde{\Box}_1$ by Lemma I.2.5.11. Thus, we only consider the case $z_n \in \mathcal{B}_5 - \tilde{\Box}_1$.

Recall $\tilde{\lambda}_2 = \inf\Big\{t > 0 \;\Big|\; Z_{2,t} \in \{c_0, 1-c_0\}\Big\}$. Observe that

$$\begin{aligned}
&\boldsymbol{1}_{\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\}}\boldsymbol{1}_{\{\tilde{\lambda}_1\leqslant 5|\ln\delta|/A\}}\\
&= \boldsymbol{1}_{\{\sup_{0\leqslant t\leqslant\tilde{\lambda}_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\}}\boldsymbol{1}_{\{\tilde{\lambda}_2\leqslant 5|\ln\delta|/A\}}\,.
\end{aligned}$$

By (I.2.5.17) and (I.2.5.21) and, we can pick n large enough such that

$$\begin{aligned}&\boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant\tilde\lambda_1}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\,;\tilde\lambda_1\leqslant\frac{5|\ln\delta|}{A}\Big)\\&\geqslant\boldsymbol{P}^{z_n}\Big(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)|\leqslant\frac{5}{\sqrt{A}}\,;\tilde\lambda_2\leqslant\frac{5|\ln\delta|}{A}\Big)\geqslant\frac{1}{2}\,.\end{aligned}$$

This is a contradiction, proving (I.2.5.20) as desired.

Step 2: Once (I.2.5.20) is established, we can estimate $\boldsymbol{E}\eta_5^\varepsilon$ as the expected time to success of a Bernoulli trial using a similar argument as in the proof of Lemma I.2.4.2. Explicitly, let $\Delta t=6|\ln\delta|/A$, and observe that by (I.2.5.20),

$$\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon<\Delta t\Big)=\boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant 6|\ln\delta|/A}|H(Z_t)|>\frac{5}{\sqrt{A}}\Big)\geqslant C\,.$$

By the strong Markov property and estimate (I.2.5.20), we have that for $i>1$,

$$\begin{aligned}\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon\geqslant i\Delta t\Big)&=\boldsymbol{E}^{z_0}\boldsymbol{E}^{z_0}\Big(\mathbf{1}_{\{\eta_5^\varepsilon\geqslant i\Delta t\}}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant(i-1)\Delta t\}}\mid\mathcal{F}_{(i-1)\Delta t}\Big)\\&=\boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant(i-1)\Delta t\}}\boldsymbol{E}^{Z_{(i-1)\Delta t}}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant\Delta t\}}\\&\leqslant\boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant(i-1)\Delta t\}}\sup_{z\in\mathcal{B}_5}\boldsymbol{E}^{z}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant\Delta t\}}\\&=\boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant(i-1)\Delta t\}}\Big(1-\inf_{z\in\mathcal{B}_5}\boldsymbol{P}^{z}\Big(\eta_5^\varepsilon<\Delta t\Big)\Big)\\&=\boldsymbol{E}^{z_0}\mathbf{1}_{\{\eta_5^\varepsilon\geqslant(i-1)\Delta t\}}(1-C)\leqslant(1-C)^i\,,\end{aligned}$$

where C is the constant in (I.2.5.20). Therefore,

$$\begin{aligned}\boldsymbol{E}^{z_0}\eta_5^\varepsilon&=\int_0^\infty\boldsymbol{P}^{z_0}(\eta_5^\varepsilon\geqslant t)\,dt\leqslant\sum_{i=1}^\infty\int_{(i-1)\Delta t}^{i\Delta t}\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon\geqslant t\Big)\,dt\\&\leqslant\Delta t\sum_{i=0}^\infty\boldsymbol{P}^{z_0}\Big(\eta_5^\varepsilon\geqslant i\Delta t\Big)\leqslant\Delta t\sum_{i=0}^\infty(1-C)^i\leqslant\frac{6|\ln\delta|}{(1-C)A}\,,\end{aligned}$$

from which (I.2.5.2) follows immediately. ☐

I.2.5.3. Proof of Lemma I.2.5.3. In this subsection, we restrict our attention to a particular cell $(0,1)\times(0,1)$ as the analysis is similar for $(1,2)\times(0,1)$. Thus, assume for simplicity that $|H|=H$. By Assumption 3, $\partial_i^2H\leqslant 0$ for $i\in\{1,2\}$. Let $z\in\overline{\mathcal{B}}_1^c$ and denote $U_\varepsilon(z)=\boldsymbol{E}^z\eta_1^\varepsilon$. Then, U_ε solves the following equation

$$\begin{cases}-\partial_1^2U_\varepsilon-\varepsilon^2\partial_2^2U_\varepsilon+Av\cdot\nabla U_\varepsilon=1&\text{in }(0,1)^2-\mathcal{B}_1\,,\\U_\varepsilon=0&\text{on }(0,1)^2\cap\partial\mathcal{B}_1\,.\end{cases}\tag{I.2.5.22}$$

In order to prove Lemma I.2.5.3, we construct an explicit supersolution to (I.2.5.22), independent of ε. Recall by Lemma I.2.4.4,

$$S \stackrel{\text{def}}{=} \sup_{\varepsilon>0} \|U_\varepsilon\|_{L^\infty} < \infty\,.$$

Let $d_1 \ll 1$ be a small constant that will be chosen later, and define

$$\Lambda = \Big\{ \frac{1}{\sqrt{A}} \leqslant |H| \leqslant d_1 \Big\}$$

$$R_2 = \Lambda \cap \{ y \in [c_0, 1-c_0] \} \quad \text{and} \quad R_1 = \Lambda - R_2\,.$$

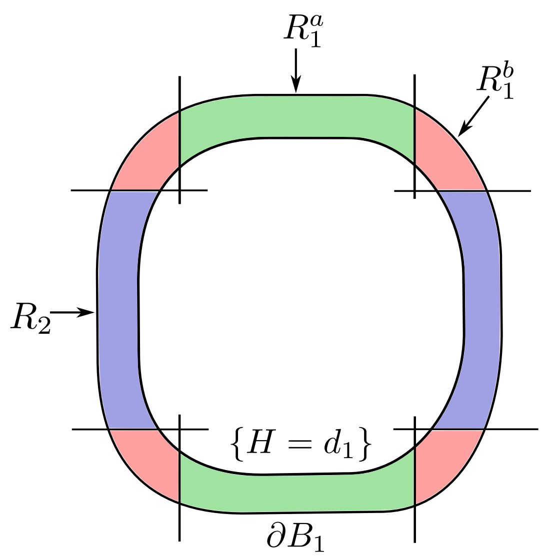

FIGURE I.2.5.2. Λ, R_1^a (green), R_1^b (red) and R_2 (blue).

Denote by (θ, h) the curvilinear coordinate, where $\theta = \Theta(x_1, x_2)$ is the “angle” and $h = H(x_1, x_2)$ the level of the Hamiltonian H (See Section I.2.6). Let f (to be specified later) be a smooth periodic function of Θ that satisfies

$$\begin{gathered} 0 < \inf f < \sup f < \infty\,, \\ -\infty < \inf f'(\Theta) \leqslant \sup f'(\Theta) < -1 \quad \text{on } R_1\,, \\ \text{and} \quad \sup |f''| < \infty\,. \end{gathered} \tag{I.2.5.23}$$

Then, consider the function

$$\phi = \chi_1 + \chi_2\,,$$

where

$$\chi_1 = -\frac{S}{d_1} H \ln H \quad \text{and} \quad \chi_2 = -\frac{f(\Theta)}{AH} + \frac{\|f\|_{L^\infty}}{\sqrt{A}}\,.$$

By construction, $\phi(\Theta, H) \geqslant 0$ on Λ. We claim that for an appropriate f, ϕ is a desired supersolution.

LEMMA I.2.5.13. *Let U_ε be the solution to equation (I.2.5.22). Then, there exists a function f that satisfies the requirement (I.2.5.23) so that for small enough d_1,*

$$\phi \geqslant U_\varepsilon \quad \text{on } \Lambda\,.$$

Postponing the proof of this lemma, we now give the proof of Lemma I.2.5.3.

PROOF OF LEMMA I.2.5.3. By construction, on $\overline{\mathcal{B}}_5 - \mathcal{B}_1$ and for small enough ε, we have $\frac{5}{\sqrt{A}} \leqslant d_1$. Therefore, when $H = 5/\sqrt{A}$,

$$\phi \leqslant -\frac{S}{d_1}\frac{5}{\sqrt{A}}\ln\Big(\frac{5}{\sqrt{A}}\Big) + \frac{\|f\|_{L^\infty}}{\sqrt{A}} \leqslant \frac{|\ln\delta|}{\sqrt{A}}\,.$$

It follows that

$$\boldsymbol{E}^z\eta_1^\varepsilon = U(z) \leqslant \phi(z) \leqslant \frac{|\ln\delta|}{\sqrt{A}}\,,$$

for every $z \in \partial\mathcal{B}_5$, as desired. ☐

PROOF OF LEMMA I.2.5.13. *Step 1:* Recall that $v = \nabla^\perp H$ and $H \geqslant 1/\sqrt{A}$. We have that

$$\nabla\chi_2 = -\frac{f'(\Theta)}{AH}\nabla\Theta + \frac{f(\Theta)}{AH^2}\nabla H\,,$$

$$\begin{aligned}
-\partial_1^2\chi_2 &= \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_1\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_1\Theta\partial_1 H + \frac{f'(\Theta)}{H}\partial_1^2\Theta\Big)\\
&\quad + \frac{1}{A}\Big(\frac{2f(\Theta)}{H^3}(\partial_1 H)^2 - \frac{f(\Theta)}{H^2}\partial_1^2 H\Big)\\
&\geqslant \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_1\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_1\Theta\partial_1 H + \frac{f'(\Theta)}{H}\partial_1^2\Theta\Big)\,,
\end{aligned}$$

and

$$-\partial_2^2\chi_2 \geqslant \frac{1}{A}\Big(\frac{f''(\Theta)}{H}(\partial_2\Theta)^2 - 2\frac{f'(\Theta)}{H^2}\partial_2\Theta\partial_2 H + \frac{f'(\Theta)}{H}\partial_2^2\Theta\Big)\,.$$

Therefore, by (I.2.5.23) and $H \geqslant 1/\sqrt{A}$,

$$-(\partial_1^2 + \varepsilon\partial_2^2)\chi_2 \geqslant -\frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\Big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\Big)\Big) - \frac{C}{\sqrt{A}}\,. \tag{I.2.5.24}$$

Step 2: On the other hand,

$$\nabla\chi_1 = -\frac{S}{d_1}(1 + \ln H)\nabla H$$

and

$$-\partial_1^2\chi_1 = \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H}$$

We note that there exists a function $\rho = \rho(x) > 0$ that

$$\nabla\Theta = \rho(x)\nabla^{\perp}H = \rho(x)v(x)\,,$$

and $\lambda_1 \leqslant \rho \leqslant \lambda_2$ on $\Big\{|H| \leqslant c_0\Big\}$ for some $0 < \lambda_1 < \lambda_2$. Therefore, by (I.2.5.24) and $H \geqslant 1/\sqrt{A}$,

$$\begin{aligned} &-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi \\ &\geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H} - \frac{f'(\Theta)|\nabla H|^2}{H}\rho \\ &\quad - \frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\Big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\Big)\Big) - \frac{C}{\sqrt{A}}\,. \end{aligned} \tag{I.2.5.25}$$

Recall

$$R_2 = \Lambda \cap \{z_2 \in [c_0, 1-c_0]\} \quad \text{ and } \quad R_1 = \Lambda - R_2\,.$$

We would like to estimate the above quantity in R_1 and R_2.

Step 3: For R_1, we decompose this set further

$$R_1^a = R_1 \cap \{c_0 \leqslant z_1 \leqslant 1 - c_0\} \quad \text{ and } \quad R_1^b = R_1 - R_1^a\,.$$

In R_1^a, there exists a constant $\tilde{C}$ such that $|\nabla H|^2 \geqslant \tilde{C}$. Therefore, by (I.2.5.23), (I.2.5.25) and $H \geqslant 1/\sqrt{A}$,

$$\begin{aligned} -\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi &\geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - C\|f'\|_{L^\infty} \\ &\geqslant \frac{\lambda_1\tilde{C}\inf_{R_1}|f'(\Theta)|}{d_1} - C\|f'\|_{L^\infty}\,. \end{aligned}$$

By (I.2.5.23), we could then pick d_1 small, independent of ε, to make the following hold

$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1$$

in R_1^a.

On the other hand, in R_1^b, we have $|\nabla H(z_1,z_2)|^2 = z_1^2 + z_2^2$. Therefore, by Cauchy-Schwarz inequality,

$$\left|f'(\Theta)\frac{|\nabla H|^2}{H}\right| = -f'(\Theta)\frac{|\nabla H|^2}{H} = -f'(\Theta)\frac{z_1^2+z_2^2}{z_1 z_2} \geqslant 2\inf_{R_1}|f'|\,. \tag{I.2.5.26}$$

Also, note that in R_1^b it holds that $|\partial_i\Theta\partial_i H| = (\partial_i H)^2$ for $i = 1, 2$. Thus, by (I.2.5.23)–(I.2.5.26) and $H \geqslant 1/\sqrt{A}$, we choose f such that $\lambda_1\inf_{R_1}|f'| > 2$ and ε small enough to get

$$\begin{aligned} &-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi \\ &\geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - \frac{2}{A}\Big(\frac{f'(\Theta)}{H^2}\Big(\partial_1\Theta\partial_1 H + \varepsilon\partial_2\Theta\partial_2 H\Big)\Big) - \frac{C}{\sqrt{A}} \end{aligned}$$

$$
\begin{aligned}
&\geqslant -\frac{f'(\Theta)|\nabla H|^2}{H}\rho - \frac{2}{A}\Big|\frac{f'(\Theta)|\nabla H|^2}{H^2}\Big| - \frac{C}{\sqrt{A}}\\
&= \Big|\frac{f'(\Theta)|\nabla H|^2}{H}\Big|\Big(\rho - \frac{2}{AH}\Big) - \frac{C}{\sqrt{A}}\\
&\geqslant \lambda_1 \inf_{R_1}|f'| - \frac{C}{\sqrt{A}} > 1\,.
\end{aligned}
$$

Thus, we have just shown that there exists a function f that satisfies (I.2.5.23) so that in R_1,

$$-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1\,.$$

Step 4: In R_2, there exist constants C_1, C_2 so that

$$0 < C_2 \leqslant C_1|\nabla H|^2 \leqslant (\partial_1 H)^2\,.$$

We then look at

$$
\begin{aligned}
&-\partial_1^2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi\\
&\quad\geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{(\partial_1 H)^2}{H} - \frac{f'(\Theta)|\nabla H|^2}{H}\rho - C\\
&\quad\geqslant \frac{S}{d_1}\partial_1^2 H(\ln H + 1) + \frac{S}{d_1}\frac{C_1|\nabla H|^2}{H} - \lambda_2\|f'\|_{L^\infty(R_2)}\frac{|\nabla H|^2}{H} - C\\
&\quad\geqslant \frac{C_2}{C_1 d_1}\Big(\frac{SC_1}{d_1} - \lambda_2\|f'\|_{L^\infty(R_2)}\Big) - C\,.
\end{aligned}
$$

Pick d_1 smaller if needed to get

$$-\partial_2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1 \quad \text{in } R_2\,.$$

Step 5: Combining Steps 3 and 4, we have shown that there exists a function f such that

$$-\partial_2\phi - \varepsilon\partial_2^2\phi + Av\cdot\nabla\phi > 1 \quad \text{in } \Lambda\,.$$

By construction, $\phi \geqslant U_\varepsilon$ on $\{H = d_1\} \cup \{H = \frac{1}{\sqrt{A}}\}$. The comparison principle then tells us that

$$\phi \geqslant U_\varepsilon \quad \text{in } \Lambda$$

as desired. □

I.2.6. Proof of Lemma I.2.4.4

In this section, we give the proof of Lemma I.2.4.4. This fact has been obtained in more generality by PDE method by Ishii and Souganidis [IS12]. Our method proof, still PDE-based, is different than that in [IS12]. Although the argument is new for our particular situation, it is an adaptation of the method in [Kum18], where the author studies the Freidlin problem for first order Hamilton-Jacobi equations.

It is convenient to work in the so-called curvilinear coordinates (h, θ), in one cell. Let $\mathcal{Q}_0^* = (0,1)^2 - \Gamma_0$, where Γ_0 is the closure of one trajectory of the gradient flow of H starting on the boundary of the unit square. On $\mathcal{Q}_0^*$ we define the curvilinear coordinates by setting $h = H(x)$, $\theta = \Theta(x)$, where Θ solves

$$\nabla\Theta \cdot \nabla H = 0\,,$$

in $\mathcal{Q}_0^*$, normalized so that the range of Θ is $(0, 2\pi)$. In this coordinate system, $h(x)$ determines the level set of the Hamiltonian to which x belongs and θ describes the position of x on this level set. Since $\nabla\Theta$ and $\nabla^\perp H$ are parallel, there must exist a non-zero function ρ such that

$$\nabla\Theta = \rho \nabla^\perp H\,.$$

By reversing the orientation of Θ if needed, we may assume, without loss of generality, that $\rho > 0$. Let $J = \partial_1 H \partial_2 \Theta - \partial_2 H \partial_1 \Theta$ be the Jacobian of the coordinate transformation, and note

$$J = \rho |\nabla H|^2\,, \quad |\nabla\Theta| = \rho |\nabla H|\,.$$

Let γ be the solution to (I.2.5.7) with $\gamma_0 = x$, and T be the time period of γ. Note T only depends on $h = H(x)$, and is given by

$$T(h) \stackrel{\text{def}}{=} \inf\{t > 0 : \gamma(t,x) = x\} = \oint_{\{H=h\}} \frac{1}{|\nabla H|}\,|d\ell|\,, \tag{I.2.6.1}$$

where $|d\ell|$ denotes the arc-length integral along the curve $\{H = h\}$.

Let $S(x) \stackrel{\text{def}}{=} \inf\{t \mid \gamma(t,x) \in \Gamma_0\}$ be the amount of time γ takes to to reach Γ_0 starting from x. This time is not a continuous function of x. Therefore, in order to make it continuous, we modify it to the following continuous function

$$\tilde{S}(x) := \begin{cases} S(x) & \text{if } S(x) > \Gamma(H(x))/2, \\ -S(x) + \Gamma(H(x)) & \text{if } S(x) < \Gamma(H(x))/2. \end{cases} \tag{I.2.6.2}$$

As we have restricted our attention to one cell, we can assume $H \in [0,1]$. Define the coefficients D_1 and D_2 on $[0,1]$ as follows

$$D_1(h) = \frac{1}{T(h)} \oint_{\{H=h\}} \frac{|\partial_1 H|^2}{|\nabla H|}\,|d\ell|\,, \tag{I.2.6.3a}$$

$$D_2(h) = \frac{1}{T(h)} \oint_{\{H=h\}} \frac{\partial_1^2 H}{|\nabla H|}\,|d\ell|\,. \tag{I.2.6.3b}$$

Note that by Gauss–Green theorem, we have

$$T(h) D_1(h) = -\int_{\{H \geqslant h\}} \partial_1^2 H(x)\,dx = \int_1^h \oint_{\{H=h\}} \frac{\partial_1^2 H}{|\nabla H|}\,|d\ell|\,dh\,.$$

Therefore,

$$\frac{d}{dh}(T(h)D_1(h)) = T(h)D_2(h)\,. \tag{I.2.6.4}$$

We are now ready to show the proof of Lemma I.2.4.4.

Proof of Lemma I.2.4.4. As before, we restrict our attention to a particular cell $(0,1)^2$ as the estimate is the same for other ones.

Step 1: Let $U_\varepsilon(x) \stackrel{\text{def}}{=} \boldsymbol{E}^x \tau_0^\varepsilon$ and $\Omega_\varepsilon \stackrel{\text{def}}{=} (0,1)^2 - \mathcal{B}_\alpha$. Then, U_ε is the solution to the equation

$$-\frac{1}{2}\partial_1^2 U_\varepsilon - \frac{\varepsilon^2}{2}\partial_2^2 U_\varepsilon + Av\cdot\nabla U_\varepsilon = 1 \quad \text{ on } \Omega_\varepsilon\,,$$

with boundary condition

$$U_\varepsilon = 0 \quad \text{ on } \partial\Omega_\varepsilon\,.$$

Lemma I.2.4.4 will follow immediately from the uniform bound

$$\sup_\varepsilon \|U_\varepsilon\|_{L^\infty(\Omega_1^\varepsilon)} \leqslant C.$$

To see why this bound is true, let us consider the solution $\bar{U}$ to the ODE

$$\begin{cases} -D_1(h)\partial_h^2\bar{U} - D_2(h)\partial_h\bar{U} & = 4\,,\\ \qquad\qquad\qquad \bar{U}(0) & = 4\,. \end{cases}$$

Note that $\bar{U}$ is bounded. To see this, we use (I.2.6.4) to rewrite the equation

$$-\frac{1}{T(h)}\partial_h\Big(T(h)D_1(h)\partial_h\bar{U}\Big) = 4\,.$$

Observe that $T(h)D_1(h) \approx O(1-h)$ and $T(h) \to T_0 > 0$ as $h \to 1$; $T(h) \approx O(|\ln h|)$ and $D_1(h) \approx O(1/|\ln h|)$ as $h \to 0$ (see Chapter 8.2 in [FW12]). Using these asymptotics, we deduce

$$\partial_h\bar{U}(h) = \frac{4}{T(h)D_1(h)}\int_h^1 T(s)\,ds\,, \qquad \bar{U}(h) = \int_0^h \frac{4}{T(s)D_1(s)}\int_s^1 T(r)\,dr ds\,,$$

and

$$\|\bar{U}\|_{W^{1,\infty}} \leqslant C\,.$$

Step 2: Note that $\bar{U}\circ H$ is a function on Ω. Let

$$g = \partial_1^2(\bar{U}\circ H)\,,$$

and we see that

$$\bar{g}(x) \stackrel{\text{def}}{=} \frac{1}{T(H(x))}\int_0^{T(H(x))} g(\gamma(t,x))\,dt = -4\,,$$

where T is defined in (I.2.6.1). Define

$$\varphi(x) = \int_0^{\tilde{S}(x)} (\bar{g}(x) - g(\gamma_t(x)))\, dt\,,$$

where $\tilde{S}$ is defined in (I.2.6.2). Note that

$$v(x) \cdot \nabla \varphi(x) = g(x) - \bar{g}(x) = g(x) + 4\,. \tag{I.2.6.5}$$

To see this, consider

$$\begin{aligned}\varphi(\gamma(s,x)) &= -\int_0^{\tilde{S}(\gamma(s,x))} \Big(g(\gamma(t, \gamma(s,x))) - \bar{g}(\gamma(s,x)) \Big)\, dt \\ &= -\int_s^{\tilde{S}(x)} \Big(g(\gamma(t,x)) - \bar{g}(x) \Big)\, dt\,.\end{aligned}$$

Differentiate in s and evaluate at $s = 0$, we get (I.2.6.5).

Step 3: Let

$$G_\varepsilon \stackrel{\text{def}}{=} \bar{U} \circ H + \frac{1}{A}\varphi\,, \qquad L_\varepsilon = -\frac{1}{2}\partial_1^2 - \frac{\varepsilon^2}{2}\partial_2^2 + Av \cdot \nabla\,,$$

and note

$$\begin{aligned}L_\varepsilon G_\varepsilon &= -\frac{1}{2}\partial_1^2(\bar{U} \circ H) - \frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U} \circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi + g(x) + 4 \\ &= -\frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U} \circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi + 4 = e_\varepsilon + 4\,,\end{aligned}$$

where $e_\varepsilon \stackrel{\text{def}}{=} -\frac{1}{2A}\partial_1^2\varphi - \frac{\varepsilon^2}{2}\partial_2^2(\bar{U} \circ H) - \frac{\varepsilon^2}{2A}\partial_2^2\varphi$. Since U is smooth and e_ε converge uniformly to 0 as $\varepsilon \to 0$, there exists an ε_0 such that for all $\varepsilon \leqslant \varepsilon_0$, $L_\varepsilon G_\varepsilon \geqslant 1$ and $G_\varepsilon \geqslant U_\varepsilon$ on $\partial\Omega_\varepsilon$. By the maximum principle, $G_\varepsilon \geqslant U_\varepsilon$ on Ω_ε. Finally, observe that $\sup_\varepsilon \|G_\varepsilon\|_{L^\infty} < \infty$, which implies what we want. $\square$

I.2.7. Upper bound for energy constrained flows

In this section our aim is to prove the upper bound in Theorem I.2.1.1. As in the proof of Proposition I.2.1.3, we will consider the doubled strip $S_2 = \mathbb{R} \times (-1, 1)$ with Dirichlet boundary conditions, and only use velocity fields v satisfying (I.2.3.1). Our aim is to find $v \in \mathcal{V}^{0,p}_{\mathscr{U}}$ satisfying (I.2.3.1) such that

$$\|T^v\|_{L^\infty} \leqslant \frac{C \ln \mathscr{U}}{\mathscr{U}}\,,$$

for all sufficiently large $\mathscr{U}$. The flow we use is an analog of the one used by Marcotte et al. [Mar+18b] adapted to the periodic strip, and is shown in Figure I.2.1.2. It consists of $1/\varepsilon$ convection rolls of width ε, height 1 skewed so that the center of the roll is only δ away from the top boundary. Here $\varepsilon, \delta > 0$ are small numbers that will shortly be chosen in terms of the Péclet number $\mathscr{U}$.

Let $\nu \in (0,1)$, $\delta = \varepsilon^{2+\nu}$ and $H\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H(x_1, x_2) \stackrel{\text{def}}{=} H_1(x_1) H_2(x_2)\,,$$

where $H_1\colon \mathbb{R} \to \mathbb{R}$, $H_2 = H_{2,\varepsilon}\colon [0,1] \to \mathbb{R}$ are Lipschitz functions such that

$$H_1(x_1 + 2) = H_1(x_1)\,, \qquad H_2(-x_2) = H_2(x_2)\,,$$

$$H_1(x_1) = \begin{cases} x_1 & x_1 \in \left[0, \dfrac{1}{2}\right)\,, \\ 1 - x_1 & x_1 \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right)\,, \\ -2 + x_1 & x_1 \in \left[\dfrac{3}{2}, 2\right)\,. \end{cases}$$

and

$$H_2(x_2) = \begin{cases} x_2 & x_2 \in [-1 + 2\delta, 1 - 2\delta]\,, \\ 0 & x_2 = \pm 1\,. \end{cases}$$

Moreover, we assume H_1, H_2 are such that H has only one non-degenerate critical point in the square $(0,2) \times (0,1)$. Stream lines of such a Hamiltonian are shown in Figure I.2.1.2.

Given $\varepsilon > 0$, define the rescaled Hamiltonian H^ε by

$$H^\varepsilon(x_1, x_2) \stackrel{\text{def}}{=} H\Big(\frac{x_1}{\varepsilon}, x_2\Big)\,, \qquad \text{and set} \quad v^\varepsilon \stackrel{\text{def}}{=} \frac{A_\varepsilon}{\varepsilon} \nabla^\perp H^\varepsilon = \frac{A_\varepsilon}{\varepsilon} \begin{pmatrix} \partial_2 H^\varepsilon \\ -\partial_1 H^\varepsilon \end{pmatrix}\,.$$

Let $T_\varepsilon = T^{v^\varepsilon}$ be the solution to (I.1.1.2)–(I.1.1.3) with drift v^ε.

By uniqueness of solutions we see that T_ε satisfies $T_\varepsilon(x_1 + 2\varepsilon, x_2) = T_\varepsilon(x_1, x_2)$. Thus, we change variables and define

$$y_1 = \frac{x_1}{\varepsilon}\,, \quad y_2 = x_2\,, \qquad \text{and} \qquad v = \nabla_y^\perp H\,.$$

In these coordinates we see that T_ε satisfies

$$A_\varepsilon v \cdot \nabla_y T_\varepsilon - \frac{1}{2}\partial_{y_1}^2 T_\varepsilon - \frac{1}{2}\varepsilon^2 \partial_{y_2}^2 T_\varepsilon = \varepsilon^2\,, \tag{I.2.7.1a}$$

with boundary conditions

$$T_\varepsilon(y_1 + 2, y_2) = T_\varepsilon(y_1, y_2)\,, \qquad \text{and} \qquad T_\varepsilon(y_1, 1) = T_\varepsilon(y_1, -1) = 0\,. \tag{I.2.7.1b}$$

To estimate the size of T_ε, consider the associated diffusion let $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$ which solves the SDE (I.2.4.1). And let τ^ε (defined in (I.2.4.2)) be the exit time of Z from the doubled strip S_2. By the Dynkin formula, we know $T_\varepsilon = \varepsilon^2 \boldsymbol{E}\tau^\varepsilon$, and so estimating $\boldsymbol{E}\tau^\varepsilon$ will give us a bound on T_ε. This is our next proposition.

PROPOSITION I.2.7.1. *Given a Hamiltonian H in the above form, choose $A_\varepsilon = 1/\varepsilon^\nu$, $v = \nabla_y^\perp H$. There exists a constant $C = C(\nu)$ such that*

$$\sup_{z \in \Omega'} \boldsymbol{E}^z \tau^\varepsilon \leqslant \frac{C|\ln \varepsilon|}{A_\varepsilon}\,. \tag{I.2.7.2}$$

for all sufficiently small ε.

The reason the bound (I.2.7.2) is as follows. In time $O(|\ln \varepsilon|/A_\varepsilon)$, deterministic trajectories of the flow v will move most interior points to $O(\delta)$ away from the $\partial_D S_2$. In this region, the drift has speed $O(A_\varepsilon/\delta)$ so particles in this region have $O(\delta/A_\varepsilon)$ time to diffuse vertically before getting carried away from the boundary $\partial_D S_2$. Within this time, particles can diffuse a vertical distance of $O(\varepsilon\sqrt{\delta/A_\varepsilon})$. By choice of $\delta = \varepsilon^2/A_\varepsilon$, and so $\varepsilon\sqrt{\delta/A_\varepsilon} = \delta$, and hence particles a distance $O(\delta)$ away from $\partial_D S_2$ exit S_2 with non-zero probability, before being carried away from $\partial_D S_2$ by the flow. Now using the strong Markov property we can estimate $\boldsymbol{E}\tau^\varepsilon$ by the expected time to success of repeated Bernoulli trials, leading to (I.2.7.2). Before carrying out these details, we first show how it can be used to finish the proof of Theorem I.2.1.1.

PROOF OF THE UPPER BOUND IN THEOREM I.2.1.1. Clearly it is enough to prove (I.2.1.4) for $q = \infty$. Let v be the flow from the Hamiltonian in Proposition I.2.7.1 and $A_\varepsilon = \varepsilon^{-\nu}$. We note that

$$\mathscr{U} = \|v^\varepsilon\|_{L^p} = O\Big(\frac{A_\varepsilon}{\varepsilon}\Big(\frac{1}{\varepsilon^p}+\frac{1}{\delta^{p-1}}\Big)^{1/p}\Big) = O\Big(\frac{A_\varepsilon}{\varepsilon}\Big(\frac{1}{\varepsilon^p}+\frac{1}{\varepsilon^{(2+\nu)(p-1)}}\Big)^{1/p}\Big) = O(\varepsilon^{-q}),$$

where

$$p' = \begin{cases} 2+\nu & 1 \leqslant p \leqslant \dfrac{2+\nu}{1+\nu}, \\[2ex] 1+\nu+\dfrac{(2+\nu)(p-1)}{p} & p \geqslant \dfrac{2+\nu}{1+\nu}. \end{cases}$$

Let T_ε be the solution to (I.2.7.1a)–(I.2.7.1b), and note that by Dynkin's formula, $T_\varepsilon = \varepsilon^2 \boldsymbol{E}\tau^\varepsilon$. Thus, by Proposition I.2.7.1

$$\|T_\varepsilon\|_{L^\infty} \leqslant \frac{C\varepsilon^2|\ln\varepsilon|}{A_\varepsilon} \leqslant \frac{C\ln\mathscr{U}}{\mathscr{U}^{(2+\nu)/p'}}.$$

If $p < 2$, then by choosing $\nu > 0$ small enough we can ensure $p \leqslant (2+\nu)/(1+\nu)$. In this case $2+\nu = p'$ and hence

$$\|T_\varepsilon\|_{L^\infty} \leqslant \frac{C\ln\mathscr{U}}{\mathscr{U}}.$$

On the other hand, if $p \geqslant 2$, then for any $\mu > 0$ we can choose $\nu > 0$ small enough to ensure

$$\|T_\varepsilon\|_{L^\infty} \leqslant \frac{C_\mu\ln\mathscr{U}}{\mathscr{U}^{\frac{2p}{3p-2}-\mu}},$$

finishing the proof. ∎

It remains to prove Proposition I.2.7.1. The key step is to show that starting from any point in S_2, the probability Z^ε hits the boundary $\partial_D S_2$ in time $O(|\ln\varepsilon|/A_\varepsilon)$ is bounded away from 0. This is our next lemma.

Lemma I.2.7.2. *Let $A_\varepsilon = \varepsilon^{-\nu}$. There exists constants $p_0 = p_0(\nu) \in (0,1)$ and $K = K(\nu) \in \mathbb{N}$, independent of ε, such that*

$$\inf_{z \in \Omega'} \boldsymbol{P}^z\Big(\tau^\varepsilon \leqslant \frac{K|\ln \varepsilon|}{A_\varepsilon}\Big) \geqslant p_0\,, \tag{I.2.7.3}$$

for all sufficiently small $\varepsilon > 0$.

Using Lemma I.2.7.2 one can prove Proposition I.2.7.1 by treating the exit from the strip as repeated Bernoulli trials.

Proof of Proposition I.2.7.1. Letting $t_i = iK|\ln \varepsilon|/A_\varepsilon$, we note

$$\begin{aligned}\sup_{z \in \Omega'} \boldsymbol{P}^z(\tau^\varepsilon \geqslant t_i) &= \sup_{z \in \Omega'} \boldsymbol{E}^z(\boldsymbol{E}^z(\mathbf{1}_{\tau^\varepsilon \geqslant t_{i-1}} \mathbf{1}_{\tau^\varepsilon \geqslant t_i} \mid \mathcal{F}_{t_{i-1}}))\\ &= \sup_{z \in \Omega'} \boldsymbol{E}^z(\mathbf{1}_{\tau^\varepsilon \geqslant t_{i-1}} \boldsymbol{P}^{Z_{t_{i-1}}}(\tau^\varepsilon \geqslant (t_i - t_{i-1})) \leqslant (1-p_0) \sup_{z \in \Omega'} \boldsymbol{P}^z(\tau^\varepsilon \geqslant t_{i-1})\,.\end{aligned}$$

and hence

$$\sup_{z \in \Omega'} \boldsymbol{P}^z(\tau^\varepsilon \geqslant t_i) \leqslant (1-p_0)^i\,.$$

Consequently,

$$\begin{aligned}\boldsymbol{E}^z \tau^\varepsilon = \int_0^\infty \boldsymbol{P}^z(\tau^\varepsilon \geqslant t)\,dt &\leqslant \sum_{i=0}^\infty (t_{i+1} - t_i) \boldsymbol{P}^z(\tau^\varepsilon \geqslant t_i)\\ &\leqslant \frac{K|\ln \varepsilon|}{A_\varepsilon} \sum_{i=0}^\infty (1-p_0)^i = \frac{K\,|\ln \varepsilon|}{p_0 A_\varepsilon}\,,\end{aligned}$$

for every $z \in \Omega'$. This yields (I.2.7.2) as desired. ∎

It remains to prove Lemma I.2.7.2, and this constitutes the bulk of this section. We will subsequently assume $A_\varepsilon = \varepsilon^{-\nu}$, and for notational convenience simply write A instead of A_ε.

Let κ_1, defined by

$$\kappa_1 \stackrel{\text{def}}{=} \inf\Big\{t \geqslant 0 \;\Big|\; Z_t^\varepsilon \in (0,2) \times (1-2\delta, 1)\Big\}\,, \tag{I.2.7.4}$$

be the first time Z_t^ε hits the set $(0,2) \times (1-2\delta,1)$.

Lemma I.2.7.3. *Let $0 < h_0 \ll c_0$ be a small constant independent of ε, and define*

$$R_{h_0} = \Omega \cap \Big(\mathcal{B}_{h_0}^c \cup (1-c_0, 1+c_0) \times (c_0, 1-c_0)\Big)\,.$$

Suppose h_0 is small enough so that $\mathcal{B}_{h_0}^c \cap (1-c_0, 1+c_0) \times (c_0, 1-c_0)$ is nonempty. There exists constants $C_0 > 0$ and $p_1 \in (0,1)$ such that

$$\inf_{z_0 \in R_{h_0}} \boldsymbol{P}^{z_0}\Big(\kappa_1 \leqslant \frac{C_0}{A}\Big) \geqslant p_1\,. \tag{I.2.7.5}$$

The proof of Lemma I.2.7.3 is based on a standard tube lemma argument and is presented in Appendix 2.A.

LEMMA I.2.7.4. *Let h_0 be as in Lemma I.2.7.3, $T_0 = \inf\{t > 0 : \gamma_{2,t} \in \{2c_0, 1-2c_0\}\}$, and $T_1 = \min\{T_0, |\ln A|/A\}$. Then*

$$\inf_{\mathcal{B}_{h_0}\cap(0,2)\times(0,c_0)} \boldsymbol{P}^{z_0}\Big(Z_{T_1} \in (1-2c_0, 1+2c_0)\times(c_0, 1-c_0)\Big) \geqslant 1-\frac{C\ln A}{A^{1/2}}\,, \tag{I.2.7.6}$$

and

$$\inf_{\mathcal{B}_{h_0}\cap(0,2)\times(1-c_0,1-2\delta)} \boldsymbol{P}^{z_0}\Big(Z_{T_1} \in \Big((0,2c_0)\cup(2-2c_0,2)\Big)\times(c_0,1-c_0)\Big) \geqslant 1-\frac{C\ln A}{A^{1/2}}\,. \tag{I.2.7.7}$$

PROOF. We only show the proof for (I.2.7.6) as (I.2.7.7) holds also by symmetry. Let $q \geqslant 2$ be some large number to be chosen later, and let $\tilde z_0$ be the point in the set $\{H \in (A^{-1/q}, h_0)\}$ which is closest to z_0. Let $\tilde d = A|z_0 - \tilde z_0|$ and γ_t be the solution to (I.2.5.7), with $\gamma_0 = \tilde z_0$. Note that, if z_0 is already in $\{H \in (A^{-1/q}, h_0)\}$, then $\tilde d = 0$. Also, by Assumption 1,

$$\frac{\tilde d}{A} \leqslant \frac{C}{A^{1/(2q)}}\,. \tag{I.2.7.8}$$

By Itô formula, we have

$$\boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant \frac{\tilde d^2}{A^2} + 2A\|v\|_{C^1}\int_0^t \boldsymbol{E}^{z_0}|Z_s-\gamma_s|^2\,ds + (1+\varepsilon^2)t\,.$$

By Gronwall's inequality, it follows that

$$\boldsymbol{E}^{z_0}|Z_t - \gamma_t|^2 \leqslant \Big(\frac{\tilde d^2}{A^2} + (1+\varepsilon^2)t\Big)e^{2\|v\|_{C^1}At}\,.$$

Now, let $T = \inf\{t > 0 \mid \gamma_{2,t} \in (2c_0, 1-2c_0)\}$, and note that $T \leqslant D\ln A/(Aq)$ for some constant $D > 0$. By (I.2.7.8), we have

$$\begin{aligned}\boldsymbol{P}^{z_0}\Big(|Z_T - \gamma_T| \geqslant \frac{c_0}{10}\Big) &\leqslant \frac{100}{c_0^2}\Big(\frac{C}{A^{2q}} + (1+\varepsilon^2)\frac{D\ln A}{Aq}\Big)e^{2\|v\|_{C^1}D\ln A/q}\\ &\leqslant CA^{2D\|v\|_{C^1}/q-1}\ln A \leqslant \frac{C\ln A}{A^{1/2}}\,,\end{aligned}$$

provided q is chosen so that $2\|v\|_{C^1}D/q - 1 < -1/2$. we have

$$\boldsymbol{P}^{z_0}\Big(|Z_T - \gamma_T| < \frac{c_0}{10}\Big) \geqslant 1 - \frac{C\ln A}{A^{1/2}}\,. \tag{I.2.7.9}$$

Since the trajectories of Z are continuous,

$$\{Z_{T_1} \in (1-2c_0, 1+2c_0)\times(c_0, 1-c_0)\} \supseteq \Big\{|Z_T-\gamma_T| < \frac{c_0}{10}\Big\}\,,$$

from which (I.2.7.6) follows. $\square$

LEMMA I.2.7.5. *There exists constants $D > 0$, $p_2 \in (0,1)$, independent of ε so that*

$$\inf_{z_0 \in \mathcal{B}_{h_0}} \boldsymbol{P}^{z_0}\Big(\kappa_1 \leqslant \frac{D|\ln A|}{A}\Big) \geqslant p_2\,. \tag{I.2.7.10}$$

PROOF. Denote

$$\begin{gathered}
\square_1 \stackrel{\text{def}}{=} (1-2c_0, 1+2c_0) \times (c_0, 1-c_0)\,,\\
\square_2 \stackrel{\text{def}}{=} \mathcal{B}_{h_0} \cap \{x_2 \in (0, c_0)\}\,,\\
\square_3 \stackrel{\text{def}}{=} \mathcal{B}_{h_0} \cap \Big((0, 2c_0) \cup (2-2c_0)\Big) \times (c_0, 1-c_0)\,,\\
\square_4 \stackrel{\text{def}}{=} \mathcal{B}_{h_0} \cap \{x_2 \in (1-c_0, 1)\}\,.
\end{gathered}$$

First, if $z_0 \in \mathcal{B}_{h_0} \cap \square_1$, we are done, by Lemma I.2.7.3.

Suppose now that $z_0 \in \square_2$. Let T_1 be as in Lemma I.2.7.4. By Lemmas I.2.7.3, I.2.7.4 and the strong Markov property we note

$$\begin{aligned}
\boldsymbol{P}^{z_0}\Big(\kappa_1 \leqslant \frac{D}{A} + T_1\Big) &\geqslant \boldsymbol{P}^{z_0}\Big(Z_{T_1} \in \square_1\Big) \inf_{z_1 \in \square_1} \boldsymbol{P}^{z_1}\Big(\kappa_1 \leqslant \frac{D}{A}\Big)\\
&\geqslant \Big(1 - \frac{C \ln A}{A}\Big) p_1\,.
\end{aligned} \tag{I.2.7.11}$$

Suppose now that $z_0 \in \square_3$. Denote $\kappa_2 \stackrel{\text{def}}{=} \inf\{t > 0 \mid Z_{1,t} \in \{2c_0, 2-2c_0\}\}$. By a similar argument as in Lemma I.2.7.4, there exists $p \in (0,1)$ such that

$$\inf_{z_0 \in \square_3} \boldsymbol{P}^{z_0}\Big(\kappa_2 \leqslant \frac{|\ln A|}{A}\Big) \geqslant p\,.$$

There are two possibilities:

(1) There exists a p_2', independent of ε such that
$$\boldsymbol{P}^{z_0}\Big(Z_{\kappa_2} \in \square_2\,; \kappa_2 \leqslant \frac{|\ln A|}{A}\Big) \geqslant p_2'\,.$$
In this case, we can apply the same argument as in (I.2.7.11) to arrive at the desired result.

(2) Otherwise, there exists a constant p_2', independent of ε such that
$$\boldsymbol{P}^{z_0}\Big(H(Z_{\kappa_2}) \geqslant h_1\,; \kappa_2 \leqslant \frac{|\ln A|}{A}\Big) \geqslant p_2'\,,$$
for some h_1 independent of ε. We can then apply Lemma I.2.7.3 to get the desired result.

The same argument works when $z_0 \in \square_4$, and this completes the proof of (I.2.7.10). $\square$

LEMMA I.2.7.6. *There exists a constant $p_3 \in (0,1)$ such that*

$$\inf_{z_0 \in \{z \mid z_2 \geqslant 1-2\delta\}} \boldsymbol{P}^{z_0}\Big(\tau^\varepsilon \leqslant \frac{\varepsilon}{A}\Big) \geqslant p_3\,. \tag{I.2.7.12}$$

PROOF. Denote $T_3(z) = \inf\{t > 0 \mid \gamma_{2,t} \leqslant 1 - 4\delta\,, \gamma_0 = z\}$, and let

$$T_4 \stackrel{\text{def}}{=} \inf_{\{z \mid z_2 \geqslant 1-2\delta\}} T_3(z)\,.$$

By definition of H we see that $T_4 \geqslant C\delta/A$ for some constant C. In time $C\delta/A$ the process Z diffuses a distance of $O(\varepsilon\sqrt{\delta/A}) = O(\delta)$ vertically, and hence should hit the top boundary with a probability that is bounded away from 0. That is, we should have

$$\boldsymbol{P}^{z_0}\Big(\tau^\varepsilon \leqslant T_4\Big) \geqslant p_3\,, \tag{I.2.7.13}$$

which immediately implies (I.2.7.12). The inequality (I.2.7.13) can proved using a tube lemma (Lemma 2.A.3) and is the same as the proof of Lemma I.2.5.9. ☐

PROOF OF LEMMA I.2.7.2. Given Lemmas I.2.7.3, I.2.7.5, I.2.7.6, the proof of (I.2.7.3) is identical to that of Lemma I.2.5.1. ☐

Appendix

2.A. Tube lemmas

In this appendix, we prove several "tube lemmas" and estimate the probability a diffusion stays close to the underlying deterministic flow. Many such estimates are standard and can be found in books (see for instance [FW12]). However, in our situation, we require estimates where the diffusion coefficient is degenerate in one direction and the amplitude of the drift is large. While the proofs follow standard techniques, the estimates themselves aren't readily available in the literature, and we present them here.

Throughout this appendix we consider the SDE

$$dZ_t = Av(Z_t)\,dt + \sigma\,dB_t\,, \tag{2.A.1}$$

where

$$\|v\|_{L^\infty} \leqslant 1\,, \quad \|Dv\|_{L^\infty} \leqslant 1\,, \tag{2.A.2}$$

$$\sigma = (\sigma_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}\,. \tag{2.A.3}$$

For notational convenience we will often denote the diagonal entries with just one subscript and write σ_i for σ_{ii} (i.e. $\sigma_1 = 1$ and $\sigma_2 = \varepsilon$).

Lemma 2.A.1. *Fix $\lambda, \beta > 0$, and define $T = T_{\beta,A}$ and $R = R_{A,\lambda}$ by*

$$T \stackrel{\text{def}}{=} \frac{\beta}{A}\,, \qquad R \stackrel{\text{def}}{=} \Big(1 - \frac{\lambda}{\sqrt{A}},\, 1 + \frac{\lambda}{\sqrt{A}}\Big) \times (1-\varepsilon,\, 1)\,. \tag{2.A.4}$$

Let $z_0 \in R$, $u \in C^1(\mathbb{R}^2)$ and let $\tilde\gamma$ be the solution to the ODE

$$\partial_t \tilde\gamma_t = Au(\tilde\gamma_t)\,dt\,, \qquad \text{with} \qquad \tilde\gamma_0 = z_0\,,$$

and $\tilde\Gamma = \{\tilde\gamma(t) \mid t \in [0,T]\}$ be the image of $\tilde\gamma$. Denote

$$L_T = \frac{A^2}{2}\int_0^T \sum_{i=1,2}\Big(\frac{|u_i(\tilde\gamma(t)) - v_i(\tilde\gamma(t))|}{\sigma_i} + \sum_{j=1}^2 \frac{\sigma_j \|\partial_j v_i\|_{L^\infty(R+\tilde\Gamma)}}{\sigma_i\sqrt{A}}\Big)^2\,dt\,.$$

Then for some $\alpha > 0$ we have

$$\boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant T}|\sigma^{-1}(Z_t - \tilde\gamma_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big) \geqslant \boldsymbol{P}\Big(\sup_{t\leqslant T}|B_t|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big)\exp\Big(-\alpha\sqrt{L_T} - \frac{1}{2}L_T\Big)$$

for all sufficiently large A. Here the notation $|z|_\infty$ denotes $\max_i|z_i|$.

Remark 2.A.2. A similar upper bound also holds, but is not needed for purposes of this paper.

PROOF. Define the process $\tilde{Z}$ by

$$d\tilde{Z}_t = Au(\tilde{\gamma}_t)\,dt + \sigma\,dB_t\,, \qquad \text{with} \qquad \tilde{Z}_0 = z_0\,.$$

Define

$$h(t) \stackrel{\text{def}}{=} A(u(\tilde{\gamma}_t) - v(\tilde{Z}_t))\,,$$

$$\hat{h}(t) \stackrel{\text{def}}{=} \sigma^{-1}h(t)\,,$$

$$M_t \stackrel{\text{def}}{=} \exp\Big(-\int_0^t \hat{h}(s)\,dB_s - \frac{1}{2}\int_0^t \hat{h}(s)^2\,ds\Big) \tag{2.A.5}$$

and a measure $\hat{\boldsymbol{P}}$ so that

$$d\hat{\boldsymbol{P}} = M_T\,d\boldsymbol{P}\,.$$

By the Girsanov theorem (see, for example, Theorem 8.6.6 in [Øks03b]), the process

$$\hat{B}_t \stackrel{\text{def}}{=} \int_0^t \hat{h}(s)\,ds + B_t$$

is a Brownian motion with respect to the measure $\hat{\boldsymbol{P}}$ up to time T. Since

$$d\tilde{Z} = Av(\tilde{Z})\,dt + \sigma\,d\hat{B}_t\,,$$

by weak uniqueness we have

$$\boldsymbol{E}^{z_0} f(Z_t) = \hat{\boldsymbol{E}}^{z_0} f(\tilde{Z}_t) = \hat{\boldsymbol{E}}^{z_0} f(\tilde{\gamma}_t + \sigma B_t) = \boldsymbol{E}^{z_0}\Big(f(\tilde{\gamma}_t + \sigma B_t)M_t\Big)\,,$$

for any test function f. Thus

$$\boldsymbol{P}^{z_0}\Big(\sup_{t\leqslant T}|\sigma^{-1}(Z_t - \tilde{\gamma}_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big) = \boldsymbol{E}^{z_0}\Big(\mathbf{1}_K M_T\Big)\,.$$

where

$$K \stackrel{\text{def}}{=} \Big\{\sup_{t\leqslant T}|B_t|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big\}\,.$$

Now let $\alpha = (2/\boldsymbol{P}^{z_0}(K))^{1/2}$, and $\hat{K}$ be the event

$$\hat{K} \stackrel{\text{def}}{=} \Big\{\Big(\int_0^T \hat{h}(t)\,dB_t\Big)^2 < \alpha^2 \int_0^T \hat{h}(t)^2\,dt\Big\}\,.$$

By Chebychev's inequality and the Itô isometry, we see

$$\boldsymbol{P}^{z_0}(\hat{K}^c) \leqslant \frac{1}{\alpha^2} = \frac{\boldsymbol{P}^{z_0}(K)}{2}\,,$$

and hence

$$\boldsymbol{P}^{z_0}(K \cap \hat{K}) \geqslant \frac{\boldsymbol{P}^{z_0}(K)}{2}\,.$$

Thus

$$\begin{aligned}\boldsymbol{E}^{z_0}(\mathbf{1}_K M_T) &\geqslant \boldsymbol{E}^{z_0}\Big(\mathbf{1}_{K\cap\hat{K}} \exp\Big(-\alpha\Big(\int_0^T \hat{h}(t)^2\,dt\Big)^{1/2} - \frac{1}{2}\int_0^T \hat{h}(t)^2\,dt\Big)\Big)\\
&\geqslant \frac{\boldsymbol{P}^{z_0}(K)}{2} \inf_K \exp\Big(-\alpha\Big(\int_0^T \hat{h}(t)^2\,dt\Big)^{1/2} - \frac{1}{2}\int_0^T \hat{h}(t)^2\,dt\Big)\,. \qquad (2.A.6)\end{aligned}$$

To estimate the exponential, note that on the event K we have

$$|\hat{h}_i(t)| = \frac{|h_i(t)|}{\sigma_i} = \frac{A}{\sigma_i}\Big|v_i(\tilde{\gamma}_t + \sigma B_t) - v_i(\tilde{\gamma}_t) + v_i(\tilde{\gamma}_t) - u_i(\tilde{\gamma}_t)\Big|$$

$$\leqslant \frac{\lambda\sqrt{A}}{\sigma_i}\sum_j \sigma_j \|\partial_j v_i\|_{L^\infty(\tilde{\Gamma}+R)} + \frac{A|u_i(\tilde{\gamma}_t) - v_i(\tilde{\gamma}_t)|}{\sigma_i}\,, \tag{2.A.7}$$

for every $i = 1, 2$. Combining (2.A.7) with (2.A.6) completes the proof. □

LEMMA 2.A.3. *Using the same notation as in Lemma 2.A.1, we now additionally assume*

$$\max_{i\in\{1,2\}} \sum_{j=1,2} \frac{\sigma_j \|\partial_j v_i\|_{L^\infty(R+\tilde{\Gamma})}}{\sigma_i} \leqslant C_0 \tag{2.A.8}$$

$$\sum_{i=1,2} \int_0^T \frac{A^2|u_i(\tilde{\gamma}_t) - v_i(\tilde{\gamma}_t)|^2}{\sigma_i^2}\,dt \leqslant C_0^2\,. \tag{2.A.9}$$

Then there exists $C_1 = C_1(C_0, \lambda, \beta) > 0$ such that

$$\boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant T} |\sigma^{-1}(Z_t - \tilde{\gamma}_t)|_\infty \leqslant \frac{\lambda}{\sqrt{A}}\Big) \geqslant C_1$$

PROOF. Following the proof of Lemma 2.A.1, and using (2.A.8)–(2.A.9) in (2.A.7) gives

$$\int_0^T |\hat{h}(t)|^2\,dt \leqslant 2C_0^2(1 + \lambda\beta d)\,.$$

Combined with (2.A.6) the lemma follows. □

Next, we show the following estimate for the side boundary layer.

LEMMA 2.A.4. *Let $z_0 \in \tilde{\mathcal{B}}_n \stackrel{\text{def}}{=} \mathcal{B}_n - [c_0, 1 - c_0] \times [0, 1]$ and $n \in \mathbb{N}$; Z_t be a stochastic process satisfying (2.A.1)–(2.A.3) and γ_t be a deterministic process satisfying*

$$\partial_t \gamma_t = Av(\gamma_t) \quad \text{with} \quad \gamma_0 = z_0\,.$$

Let T, R be as in (2.A.4), and $\Gamma = \{\gamma(t) \mid t \in [0, T]\}$ be the image of γ, and assume

$$\partial_1 v_2 = 0 \qquad \text{in } \Gamma + R\,. \tag{2.A.10}$$

For $M \geqslant 1$, let $\tilde{R}_\varepsilon \subseteq [1 - M/\sqrt{A}, 1 + M/\sqrt{A}]$ be a Borel set, and $T = m/A$ for some $m \in \mathbb{N}$. Then, there exists a constant $C = C_{m,M}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\begin{aligned}&\boldsymbol{P}^{z_0}\Big(\sup_{0\leqslant t\leqslant T} |Z_{1,t} - \gamma_{1,t}| \leqslant \frac{2M}{\sqrt{A}}\,, \sup_{0\leqslant t\leqslant T} |Z_{2,t} - \gamma_{2,t}| \leqslant \frac{\varepsilon}{\sqrt{A}}\,, Z_{1,T} - \gamma_{1,T} \in \tilde{R}_\varepsilon\Big)\\ &\qquad \geqslant C_{m,n}\boldsymbol{P}\Big(|B_t| \leqslant \frac{2M}{\sqrt{A}}\,, B_{1,T} \in \tilde{R}_\varepsilon\Big)\end{aligned} \tag{2.A.11}$$

As before we write $Z = (Z_1, Z_2)$, $\gamma = (\gamma_1, \gamma_2)$, and the notation $Z_{i,t}$ and $\gamma_{i,t}$ denotes the values of the coordinate processes Z_i and γ_i respectively at time t.

PROOF. We follow the proof of Lemma 2.A.1, and explicitly substitute $\sigma_1 = 1$ and $\sigma_2 = \varepsilon$. Our conclusion (2.A.11) will follow provided we can show

$$\int_0^T \hat{h}(t)^2 \, dt \leqslant C \,, \tag{2.A.12}$$

for some finite constant C, independent of ε. To bound this, we use the upper bound (2.A.7), and observe that the second term on the right hand side is identically 0 since $u = v$. For the first term, the only term that may grow faster than $\sqrt{A}$ is when $i = 2$ and $j = 1$. In this case, the assumption (2.A.10) guarantees that this term is identically 0. Now squaring and integrating from 0 to $T = m/A$ proves (2.A.12) as desired. □

REMARK 2.A.5. If the velocity field v does not satisfy (2.A.10), then Lemma 2.A.4 still holds provided A is chosen so that $A \geqslant 1/\varepsilon^2$. To see this we note that (2.A.7) implies

$$\int_0^T \hat{h}(t)^2 \, dt \leqslant \frac{Cm}{A\varepsilon^2} \,.$$

If $A \geqslant 1/\varepsilon^2$ the right hand side of this is bounded independent of ε, and so the remainder of the proof of Lemma 2.A.4 remains unchanged.

Finally, we prove Lemmas I.2.7.3, and Lemma I.2.7.3, which were used in the proofs of Theorem I.2.1.1 and Proposition I.2.1.3. Both proofs follow along the lines of the above tube lemmas.

PROOF OF LEMMA I.2.5.4. We only consider the case where $z_0 \in Q_0/2$. The other cases are similar. First, recall that, by a direct calculation, we can check $T \leqslant |\ln \delta|/A$. Therefore, for small enough ε, under the event $\{|Z_{i,t} - \gamma_{i,t}| \leqslant \sigma_i(|\ln \delta| A)^{-1/2} \,, \ \forall t \leqslant T \,, i = 1, 2\}$, we must have $Z_t \in Q_0$ for $t \leqslant T$. Thus,

$$v_1(Z_t) = Z_{1,t} \quad \text{and} \quad v_2(Z_t) = -Z_{2,t} \,. \tag{2.A.13}$$

Now define

$$d\tilde{Z}_t = A \begin{pmatrix} v_1(\gamma_t) \\ v_2(\gamma_t) \end{pmatrix} dt + \sigma \, dB_t$$

and write

$$h(t) \stackrel{\text{def}}{=} A \begin{pmatrix} v_1(\gamma_t) - v_1(\tilde{Z}_t) \\ v_2(\gamma_t) - v_2(\tilde{Z}_t) \end{pmatrix} = A \begin{pmatrix} \gamma_{1,t} - \tilde{Z}_{1,t} \\ -\gamma_{2,t} + \tilde{Z}_{2,t} \end{pmatrix} = A \begin{pmatrix} -B_{1,t} \\ \varepsilon B_{2,t} \end{pmatrix} . \tag{2.A.14}$$

As before, we define $\hat{h}$ and a new measure $\hat{\boldsymbol{P}}$ by

$$\hat{h}(t) \stackrel{\text{def}}{=} \sigma^{-1} h(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix} h(t) = A \begin{pmatrix} -B_{1,t} \\ B_{2,t} \end{pmatrix} ,$$

$$d\hat{\boldsymbol{P}} = M_T \, d\boldsymbol{P} \,,$$

where

$$M_t \stackrel{\text{def}}{=} \exp\Big(- \int_0^t \hat{h}(s) \, dB_s - \frac{1}{2} \int_0^t \hat{h}(s)^2 \, ds \Big) \,,$$

for $0 \leqslant t \leqslant T$. By the Girsanov theorem, the process

$$\hat{B}_t \stackrel{\text{def}}{=} \int_0^t \hat{h}(s) \, ds + B_t$$

is a Brownian motion with respect to the measure $\hat{\boldsymbol{P}}$. Therefore, by uniqueness of weak solutions of SDEs, we have

$$\begin{aligned} \boldsymbol{E}(f(Z_t)) = \hat{\boldsymbol{E}}(f(\tilde{Z}_t)) &= \hat{\boldsymbol{E}}(f(\gamma_{1,t} + B_{1,t}, \gamma_{2,t} + \varepsilon B_{2,t})) \\ &= \boldsymbol{E}(f(\gamma_{1,t} + B_{1,t}, \gamma_{2,t} + \varepsilon B_{2,t}) M_t) \,. \end{aligned}$$

Hence

$$\begin{aligned} &\boldsymbol{P}^x \Big(|Z_{i,t} - \gamma_{i,t}| \leqslant \frac{\sigma_i}{\sqrt{|\ln \delta| A}} \,,\ \forall t \leqslant T \,, i = 1, 2 \Big) \\ &\qquad = \boldsymbol{E}^x \Big(\mathbf{1}_{\left\{ |B_t|_\infty \leqslant (|\ln \delta| A)^{-1/2} \,,\ \forall t \leqslant T \right\}} M_T \Big) \,. \end{aligned}$$

Now, we have that, by Itô formula,

$$\begin{aligned} \int_0^t \hat{h}(s) \, dB_s &= -A \int_0^t B_{1,s} \, dB_{1,s} + A \int_0^t B_{2,s} \, dB_{2,s} \\ &= \frac{A}{2} (-B_{1,t}^2 + B_{2,t}^2) \,. \end{aligned}$$

Therefore,

$$M_t \geqslant \exp\Big(-\frac{A}{2} (B_{1,t}^2 + B_{2,t}^2) - A^2 \int_0^t (B_{1,s}^2 + B_{2,s}^2) \, ds \Big) \,.$$

Therefore, as $T \leqslant |\ln \delta|/A$, under the event

$$K \stackrel{\text{def}}{=} \Big\{ |B_t|_\infty \leqslant \frac{1}{\sqrt{|\ln \delta| A}} \,,\ \forall t \leqslant T \Big\} \,,$$

we must have

$$M_T \geqslant \exp\Big(- \frac{1}{2 |\ln \delta|} - 2 \Big) \geqslant C \,.$$

Since $\boldsymbol{P}(K) \approx 1/|\ln \delta|^2$, this finishes the proof. □

Proof of Lemma I.2.7.3. Let $z_0 \in R_{h_0}$ and $T_0 = \inf\{ t > 0 \,|\, \gamma_{2,t} \geqslant 1 - \delta \}$, where γ is the solution to (I.2.5.7) with $\gamma_0 = z_0$. A direct calculation shows that there exists C_0 for which $T_0 \leqslant C_0/A$. Furthermore, when $x_2 \in (0, 1 - 2\delta)$, we have that

$$v(x_1, x_2) = \begin{pmatrix} \partial_2 H(x) \\ \partial_1 H \end{pmatrix} = \begin{pmatrix} H_1(x_1) \\ \pm x_2 \end{pmatrix} \,.$$

Therefore, following the proof of the tube lemma (Lemma 2.A.1), we find that the function $\hat{h}(t)$ there satisfies

$$|\hat{h}(t)| = A \begin{pmatrix} |H_1(\gamma_{1,t}) - H_1(\gamma_{1,t} + B_{1,t})| \\ |B_{2,t}| \end{pmatrix} .$$

Therefore, under the event $\left\{\sup_{t \leqslant T_0} |B_t| \leqslant \sqrt{T_0}\,; B_{2,T_0} \geqslant 0\right\}$, it is true that

$$\int_0^{T_0} |\hat{h}(t)|^2 \, dt \leqslant C \,. \tag{2.A.15}$$

We have that

$$K_1 \stackrel{\text{def}}{=} \left\{\sup_{t \leqslant T_0} |Z_t - \gamma_t| \leqslant \sqrt{T_0}\,; Z_{2,T_0} \geqslant 1 - 2\delta\right\} \subseteq \left\{\kappa_1 \leqslant \frac{C_0}{A}\right\} .$$

Following the proof of Lemma 2.A.1, by Girsanov's theorem and (2.A.15), there exists $p_1 \in (0,1)$ such that

$$\boldsymbol{P}^{z_0}(K_1) \geqslant C \boldsymbol{P}\left(\sup_{t \leqslant T_0} |B_t| \leqslant \sqrt{T_0}\,; B_{2,T_0} \geqslant 0\right) \geqslant p_1 \,,$$

from which (I.2.7.5) follows immediately. □

Part II

Coagulation-Fragmentation equations

CHAPTER 1

Introduction

The Coagulation-Fragmentation equation (C-F) is an integrodifferential equation that finds applications in many different fields, ranging from astronomy to polymerization to the study of animal group sizes. The equation, with pure coagulation, dates back to Smoluchowski [Smo16a], when he studied the evolution of number density of particles as they coagulate. Later on, Blatz and Tobolsky [BT45] use the full C-F to study polymerization-depolymerization phenomena. The mathematical studies of this equation did not start until the work of Melzak [Mel57], which was concerned with existence and uniqueness of the solutions for bounded kernels. Since then, although there are still a lot of open questions remain, major advancing has been made by both analytic and probabilistic tools. We list here some, but not exhaustive, important works that are relevant to our work. For existence and uniqueness of solutions, there are the works of McLeod [McL62], Ball and Carr [BC90], Norris [Nor99], Escobedo, Laurençot, Mischler and Perthame [EMP02; Esc+03]. For large time behavior of solutions, there are the works of Aizenman and Bak [AB79], Cañizo [Cañ07], Carr [Car92], Menon and Pego [MP04; MP06; MP08], Degond, Liu and Pego [DLP17], Liu, Niethammer and Pego [LNP19], Niethammer and Velázquez [NV13] and Laurençot [Lau19a]. For surveys of what has been done, we refer the readers to two dated by now but still excellent surveys by Aldous [Ald99] and da Costa [dCos15] and the new monographs by Banasiak, Lamb, and Laurençot [BLL19].

Here, coagulation represents binary merging when two clusters of particles meet, which happens at some pre-determined rates; and fragmentation represents binary splitting of a cluster, also at some pre-determined rates. Thus, the C-F describes the evolution of cluster sizes over time given that there are only coagulation and fragmentation that govern the dynamics.

A particularly interesting phenomenon of the C-F is that given the right conditions, the solution, while still physical, does not conserve mass at all time. There are two ways that this could happen. One comes from the formation of particles of infinite size; the other comes from the formation of particles of size zero, both in finite time. The first, called *gelation*, happens when the coagulation is strong enough [Esc+03]. The latter, called *dust formation*, happens when the fragmentation is strong enough (see Bertoin [Ber06]). Typically, these phenomena happen depending on the relative strengths between the coagulation kernel and fragmentation kernel, not so much on the initial data. However, there are borderline situations, where it is not very clear how solutions would behave, hence more careful analysis needs to be done based on initial data.

Both are very interesting and rich phenomena, and have been studied in various contexts.

This part of the thesis is devoted to answer some of the questions concerning the well-posedness and dynamics of solutions of the C-F in the regimes where the phenomenon of gelation plays interesting roles. Our main approach is to develop new techniques to analyze equations that result from the so-call Bernstein transform, applied to the C-F. The use of Bernstein transform to study the C-F for certain kernels was pioneered by Menon and Pego [MP04].

The results presented in this part of the thesis are from the works of the author with Tran [TV21] (Chapter 2) and Pego [PV21] (Chapter 3).

II.1.1. The coagulation-fragmentation equation

To mathematically describe the C-F, we let $c(s,t) \geqslant 0$ be the density of clusters of particles of size $s \geqslant 0$ at time $t \geqslant 0$. The evolution c is then given by

$$\partial_t c = Q_c(c) + Q_f(c)\,. \tag{II.1.1.1}$$

Here, the coagulation term Q_c and the fragmentation term Q_f could descibe discrete dynamics (over $\mathbb{N}$) or continuous dynamics (over $[0,\infty)$). In the discrete case,

$$Q_c(c)(k,t) = \frac{1}{2}\sum_{j=1}^{k-1} a(k-j,j)c(k-j,t)c(j,t) - c(k,t)\sum_{j=1}^{\infty} a(k,j)c(j,t)\,,$$

and

$$Q_f(c)(k,t) = -\frac{1}{2}c(k,t)\sum_{j=1}^{k-1} b(k-j,j) + \sum_{j=1}^{\infty} b(k,j)c(k+j,t)\,.$$

Analogously, in the continuous case,

$$Q_c(c)(s,t) = \frac{1}{2}\int_0^s a(y,s-y)c(y,t)c(s-y,t)\,dy - c(s,t)\int_0^\infty a(s,y)c(y,t)\,dy\,,$$

and

$$Q_f(c)(s,t) = -\frac{1}{2}c(s,t)\int_0^s b(s-y,y)\,dy + \int_0^\infty b(s,y)c(y+s,t)\,dy\,.$$

In both cases, a and b are called the coagulation kernel and fragmentation kernel, respectively. They are nonnegative and symmetric functions defined on $\mathbb{N}^2$ or $[0,\infty)^2$. We will specify these kernels later in the subsequent chapters.

Weak solution. We say that $c : \mathbb{N} \to [0,\infty)$ is a weak solution to the discrete coagulation-fragmentation if for every bounded function $\phi : \mathbb{N} \to [0,\infty)$, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{k=1}^{\infty}\phi(k)c(k,t)\\
&= \frac{1}{2}\sum_{j,k=1}^{\infty}(\phi(j+k)-\phi(j)-\phi(\hat{k}))a(j,k)c(j,t)c(k,t)\\
&\quad -\frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}(\phi(k)-\phi(j)-\phi(k-j))b(j,k-j)c(k,t)\,.
\end{aligned} \tag{II.1.1.2}$$

Likewise, we say that $c : [0,\infty)^2 \to [0,\infty)$ is a weak solution to the continuous coagulation-fragmentation equation if for every test function $\phi \in BC([0,\infty)) \cap \mathrm{Lip}([0,\infty))$ with $\phi(0) = 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^{\infty}\phi(s)c(s,t)\,ds\\
&= \frac{1}{2}\int_0^{\infty}\int_0^{\infty}(\phi(s+\hat{s})-\phi(s)-\phi(\hat{s}))a(s,\hat{s})c(s,t)c(\hat{s},t)\,d\hat{s}ds\\
&\quad -\frac{1}{2}\int_0^{\infty}\left(\int_0^{s}(\phi(s)-\phi(\hat{s})-\phi(s-\hat{s}))\,d\hat{s}\right)b(\hat{s},s-\hat{s})c(s,t)\,ds\,.
\end{aligned} \tag{II.1.1.3}$$

Here, $BC([0,\infty))$ is the class of bounded continuous functions on $[0,\infty)$, and $\mathrm{Lip}([0,\infty))$ is the class of Lipschitz continuous functions on $[0,\infty)$.

II.1.2. Some mathematical tools

In this section, we remind the reader about some main tools that we will employ later on. Even though these tools are somewhat new to the study of C-F, they are quite standard in other parts of mathematics and have been developed a lot in the last few decades.

II.1.2.1. Bernstein transform and functions. In this subsection, we record a representation theorem of Bernstein functions. The results here are classical and can be found in great details in the book by Schilling, Song and Vondraček [SSV12].

Definition II.1.2.1. A function $f : (0,\infty) \to [0,\infty)$ is a Bernstein function if $f \in C^{\infty}((0,\infty))$ and, for $n \in \mathbb{N}$,

$$(-1)^{n+1}\frac{d^n}{dx^n}f \geqslant 0\,.$$

THEOREM II.1.2.2. *A function $f : (0,\infty) \to [0,\infty)$ is a Bernstein function if and only if it can be written (uniquely) as*

$$f(x) = a_0 x + a_\infty + \int_{(0,\infty)} (1 - e^{-sx})\mu(ds), \qquad x \in (0,\infty)\,, \tag{II.1.2.1}$$

where $a_0, a_\infty \geqslant 0$ and μ is a measure such that

$$\int_{(0,\infty)} \min\{1, s\}\, \mu(ds) < \infty\,.$$

The triple (a_0, a_∞, μ) is called the Lévy triple.

In other words, a Bernstein function is a Bernstein transform on the extended real line $[0,\infty]$. The proof of this theorem and more beautiful properties of Bernstein functions and transform could be found in the book by Schilling, Song, and Vondraček [SSV12].

Next, consider $f : [0,\infty) \to [0,\infty)$ which is a Bernstein function such that $f(0) = 0$ and f is sublinear. By Theorem II.1.2.2, f has the representation formula (II.1.2.1). Firstly, let $x \to 0^+$ to get that

$$a_\infty = \lim_{x\to 0^+} f(x) = 0\,.$$

Secondly, divide (II.1.2.1) by x, let $x \to \infty$ and use the sublinearity of F to yield further that

$$a_0 = \lim_{x\to\infty} \frac{f(x)}{x} = 0\,.$$

Thus, under two additional conditions that $f(0) = 0$ and f is sublinear, we get that $a_0 = a_\infty = 0$, and therefore,

$$f(x) = \int_{(0,\infty)} (1 - e^{-sx})\mu(ds), \qquad x \in (0,\infty)\,.$$

For each Lévy triple (a_0, a_∞, μ), we can define a finite measure κ on $[0,\infty]$ by

$$d\kappa(x) = a_0 d\delta_0(x) + a_\infty d\delta_\infty(x) + \min\{x, 1\}\, d\mu(x)\,. \tag{II.1.2.2}$$

The Bernstein transform is also known as the "Laplace exponent" in probability literature. It is a more general notion of the Laplace transform as it can be used to deal with certain singular measures near 0. We note that the derivative of a Bernstein transform of a measure is completely monotone and, hence, is a Laplace transform of a measure. The following continuity theorem is useful when one has to deal with limits of Bernstein transforms.

THEOREM II.1.2.3 ([ILP18]). *Let $(a_0^k, a_\infty^k, \mu^k)$ be a sequence of Lévy triples with Bernstein transforms f^k (defined by (II.1.2.1)) and the associated measures κ^k (defined by (II.1.2.2)). Then the following statements are equivalent:*

(1) The function $f(x) \overset{\text{def}}{=} \lim_{k\to\infty} f^k(x)$ exists for each $x \in (0,\infty)$.

(2) The sequence κ^k converges weakly-$\star$ on $[0,\infty]$ to some finite measure κ. That is, for every $g \in C([0,\infty])$, we have

$$\lim_{k\to\infty} \langle g, \kappa^k \rangle = \langle g, \kappa \rangle \,.$$

If either condition holds then f and κ corresponds to a unique Lévy triple via (II.1.2.1) *and* (II.1.2.2)*, respectively.*

To our knowledge, the proof of this theorem is surprisingly recent [MP08].A simplified proof is given in the appendix of the paper [ILP18].

II.1.2.2. Viscosity solutions. In this subsection, we record the definition and a few facts about viscosity solutions for first order Hamilton-Jacobi equations. The results here are classical and can be found in great details in the books by Bardi, Capuzzo-Dolcetta and Crandall [BC97; Cra97].

Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. Consider the Cauchy problem for following Hamilton-Jacobi equation on $\Omega \times [0,\infty)$

$$\begin{cases} \partial_t F + H(x, F, DF) = 0 & x \in \Omega\,, \\ F(x,t) = G(x,t) & x \in \partial\Omega\,, \\ F(x,0) = F_0(x)\,. \end{cases} \tag{II.1.2.3}$$

Definition II.1.2.4. For each $T > 0$, a function $F : \Omega \times [0,T) \to \mathbb{R}$ is called:

(a) a viscosity sub-solution of (II.1.2.3) if $F \in \mathrm{USC}(\Omega \times [0,T))$, $F(\cdot,0) \leqslant F_0$, $F(\cdot,t) \leqslant G$ on $\partial\Omega$, and for every $\varphi \in C^1(\Omega \times (0,T))$ such that $F(x_0,t_0) = \varphi(x_0,t_0)$ and $F - \varphi$ has a strict max at (x_0,t_0), then

$$\partial_t \varphi(x_0,t_0) + H(x, \varphi(x_0,t_0), D\varphi(x_0,t_0)) \leqslant 0\,.$$

(b) a viscosity super-solution of (II.1.2.3) if $F \in \mathrm{LSC}(\Omega \times [0,T))$, $F(\cdot,0) \geqslant F_0$, $F(\cdot,t) \geqslant G$ on $\partial\Omega$, and for every $\varphi \in C^1(\Omega \times (0,T))$ such that $F(x_0,t_0) = \varphi(x_0,t_0)$ and $F - \varphi$ has a strict min at (x_0,t_0), then

$$\partial_t \varphi(x_0,t_0) + H(x, \varphi(x_0,t_0), D\varphi(x_0,t_0)) \geqslant 0\,.$$

(c) a viscosity solution of II.1.2.3 if it is both a viscosity sub-solution and a viscosity super-solution.

For the purposes of this thesis, we suppose that the Hamiltonian $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions

$$\begin{cases} H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R)) \quad \forall R > 0\,, \\ \lim\limits_{|p|\to\infty} \inf\limits_{x\in\mathbb{R}^n} H(x,p) = \infty\,. \end{cases} \tag{H}$$

For unbounded domain, solutions to equation (II.1.2.3) could be non-unique if there are no extra criteria to select the right solution. One such criterion could be sublinearity, which is relevant to our work in this thesis. More general discussions about the selection criteria on unbounded domains could be found in [BC97; Cra97].

THEOREM II.1.2.5 (Comparison principle). *Let* $\Omega = [0, \infty)$ *and assume condition* (H). *Suppose* F^* *and* F_* *are super-solution and sub-solution, respectively, to the equation* (II.1.2.3) *such that*

$$\lim_{x \to \infty} \frac{F^*}{x} = \lim_{x \to \infty} \frac{F_*}{x} = 0\,.$$

Then $F_* \leqslant F^*$.

COROLLARY II.1.2.6. *Let* $\Omega = [0, \infty)$ *and assume condition* (H). *Then, there is at most one sub-linear solution to equation* (II.1.2.3).

REMARK II.1.2.7. After having the comparison principle, to show existence, one will need to construct a sub-solution and a super-solution that has the desired property (in this case, sublinearity) and then use Perron's method to conclude. This is the task for our specific problem in Chapter 2.

CHAPTER 2

Well-posedness for multiplicative coagulation and constant fragmentation kernels

II.2.1. Introduction

The main goal of this chapter is to propose a new framework to analyze a borderline situation described by Escobedo, Laurençot, Mischler and Perthame [EMP02; EMP02], where solutions to the C-F may or may not exhibit gelation, depending on the initial data (as opposed to the type of kernels). In particular, we analyze the properties of viscosity solutions of a new singular Hamilton-Jacobi equation (H-J), which results from transforming the C-F equation via the so-called Bernstein transform. This, in our opinion, is natural and elegant since it requires very minimal assumptions.

Throughout this chapter, we only deal with the continuous C-F and always assume that

$$a(s,\hat{s}) = s\hat{s} \quad \text{and} \quad b(s,\hat{s}) = 1 \quad \text{for all } s,\hat{s} > 0\,.$$

II.2.1.1. The Bernstein transform. Consider the Bernstein transform of c, for $(x,t) \in [0,\infty)^2$,

$$F(x,t) \stackrel{\text{def}}{=} \int_0^\infty (1-e^{-sx})c(s,t)\,ds$$

and let

$$\phi_x(s) = 1 - e^{-sx}\,,$$

we have

$$\begin{aligned}
\partial_t F(x,t) &= \frac{1}{2}\int_0^\infty \int_0^\infty (1 - e^{-(s+\hat{s})x} - 1 + e^{-sx} - 1 + e^{-\hat{s}x}) s c(s,t)\hat{c}(\hat{s},t)\,d\hat{s}ds \\
&- \frac{1}{2}\int_0^\infty \int_0^s (1 - e^{-sx} - 1 + e^{-(s-\hat{s})x} - 1 + e^{-\hat{s}x})\,d\hat{s}\,c(s,t)\,ds \\
&= -\frac{1}{2}\int_0^\infty \int_0^\infty (1-e^{-sx})(1-e^{-\hat{s}x}) s c(s,t)\hat{s}c(\hat{s},t)\,d\hat{s}ds \\
&- \frac{1}{2}\int_0^\infty (-s - se^{-sx} + \frac{2}{x}(1-e^{-sx}))c(s,t)\,ds \\
&= -\frac{1}{2}(m_1(t) - \partial_x F(x,t))^2 + \frac{m_1(t)}{2} + \frac{\partial_x F(x,t)}{2} - \frac{F(x,t)}{x} \\
&= -\frac{1}{2}(m_1(t) - \partial_x F(x,t))(m_1(t) - \partial_x F(x,t) + 1) - \frac{F(x,t)}{x} + m_1(t)\,.
\end{aligned}$$

Here, $m_1(t)$ is the total mass (first moment) of all particles at time $t \geqslant 0$, that is,

$$m_1(t) = \int_0^\infty sc(s,t)\,ds\,.$$

Let us assume that $m_1(t) < \infty$ for all $t \geqslant 0$. The key point is to transform a seemingly hopeless nonlocal equation to a somewhat more tractable nonlinear PDE, which enjoys some major developments in the past few decades. If conservation of mass holds, then we can assume $m_1(t) = m > 0$ for all $t \geqslant 0$ for some $m \in (0,\infty)$. This fact, together with the above computations, leads to the following PDE for F.

$$\partial_t F + \frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{x} - m = 0 \quad \text{in } (0,\infty)^2\,, \tag{II.2.1.1a}$$

$$0 \leqslant F(x,t) \leqslant mx \quad \text{on } [0,\infty)^2\,, \tag{II.2.1.1b}$$

$$F(x,0) = F_0(x) \quad \text{on } [0,\infty)\,. \tag{II.2.1.1c}$$

One then can study wellposedness and properties of solutions of (II.2.1.1) to deduce back information of C-F. Indeed, this is our main goal.

Note that the condition (II.2.1.1b) implies that $F(0,t) = 0$ and that it comes directly from the Bernstein transform. Indeed, as $c \geqslant 0$, it is clear that $F \geqslant 0$. Besides, the inequality $1 - e^{-sx} \leqslant sx$ for $s, x \geqslant 0$ gives

$$F(x,t) = \int_0^\infty (1 - e^{-sx})c(s,t)\,ds \leqslant \int_0^\infty sxc(s,t)\,ds = mx\,.$$

Moreover, the dominated convergence theorem gives

$$\lim_{x\to\infty} \frac{F(x,t)}{x} = \lim_{x\to\infty} \int_0^\infty \frac{1 - e^{-sx}}{x} c(s,t)\,ds = 0\,,$$

which means that $F(x,t)$ is sublinear in x. Here, for a given function $\psi : [0,\infty) \to \mathbb{R}$, we say that it is sublinear if

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 0.$$

It is therefore natural to search for solutions of (II.2.1.1) that are sublinear in x.

It is worth noting that (II.2.1.1) is a Hamilton-Jacobi equation with the Hamiltonian

$$H(p,z,x) = \frac{1}{2}(p-m)(p-m-1) + \frac{z}{x} - m \quad \text{for all } (p,z,x) \in \mathbb{R}\times\mathbb{R}\times(0,\infty)\,,$$

which is of course singular at $x = 0$. Besides, H is monotone, but not Lipschitz in z as

$$\partial_z H(p,z,x) = \frac{1}{x} \geqslant 0 \quad \text{and} \quad \lim_{x\to 0+} \partial_z H(p,z,x) = \lim_{x\to 0+} \frac{1}{x} = +\infty\,.$$

This means that (II.2.1.1) does not fall into the classical theory of viscosity solutions to Hamilton-Jacobi equations developed by Crandall and Lions [CL83] (see also Crandall, Evans and Lions [CEL84]). It is thus our purpose to develop a framework to study wellposedness and further properties of solutions to (II.2.1.1). For a different class of Hamilton-Jacobi equations that is singular in p (but not in z), see the radially symmetric setting in Giga, Mitake and Tran [GMT16].

We emphasize that for wellposedness and regularity results, we do not need to impose all the properties of the Bernstein transform of the initial data $c_0 = c(\cdot, 0)$. To be precise, a Bernstein transform of a measure is a $C^\infty((0,\infty))$ (in fact, analytic) function. However, we only assume F_0 to be Lipschitz and sublinear for our wellposedness result and more regular for our regularity results.

A more important point is that our assumption on c_0 is minimal. For existence and uniqueness results, we do not have any restrictions on moments of c_0 except finite first moment so that the derivative of the Bernstein transform makes sense. In particular, we only require

$$m_1(0) = \int_0^\infty s c_0(s)\, ds < \infty\,.$$

This also makes physical sense since one often wishes that the initial total mass to be finite before talking about conservation of mass. Of course, we will need to put in more conditions for our regularity results.

Remark II.2.1.1. In fact, we are also able to define weak solutions in the measure sense to (II.1.1.1) in a similar fashion.

For each $t \geqslant 0$, let $c_t(ds)$ be a positive Radon measure in $(0,\infty)$. Then, we say that $c_t(ds)$ is a weak solution in the measure sense to (II.1.1.1) if for every test function $\phi \in BC([0,\infty)) \cap \mathrm{Lip}([0,\infty))$ with $\phi(0) = 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \phi(s)\, c_t(ds) = \frac{1}{2}\int_0^\infty\int_0^\infty (\phi(s+\hat{s}) - \phi(s) - \phi(\hat{s}))s\hat{s}\, c_t(ds)c_t(d\hat{s}) \\ - \frac{1}{2}\int_0^\infty \left(\int_0^s (\phi(s) - \phi(\hat{s}) - \phi(s-\hat{s}))\, d\hat{s}\right) c_t(ds)\,.\end{aligned}$$

This is clearly a weaker notion of solutions than that in (II.1.1.3). Nevertheless, the Bernstein transform of $c_t(ds)$ and (II.2.1.1) still make perfect sense. We will use this notion of solutions when talking about the existence results for the C-F.

II.2.1.2. A conjecture. In [EMP02; Esc+03], the authors conjectured that in borderline situations where coagulation kernel and fragmentation kernel balance each other out, the solution will conserve mass if the initial data have small enough total mass. Otherwise, for large total mass initial data, gelation will occur. In the paper by Vigil and Ziff [VZ89], the authors argued that if the zeroth moment of the solution reaches negative value in finite time, one expects coagulation to dominate, hence gelation will occur.

It has been expected by experts in the field that for our specific kernels, the critical initial mass should be $m_1(0) = 1$ so that for $m_1(0) > 1$, one has gelation; and for $m_1(0) \leqslant 1$, one has solutions that conserve mass. We give here a simple reason why such expectation arises.

Integrating equation (II.1.1.1) and denoting $m_0(t) = \int_0^\infty c(s,t)\,ds$, the zeroth moment, we get the following equation

$$\frac{d}{dt}m_0(t) = \frac{1}{2}m_1(t)(1 - m_1(t))\,.$$

Suppose now $m_1(t) = m_1(0) > 1$ as it is true before gelation occurs (if ever). Then $m_0(t)$ will be negative in finite time. On the other hand, $m_0(t)$ remains positive if $0 \leqslant m_1(0) \leqslant 1$. Therefore, by the reasoning above, $m_1(0) = 1$ is believed to be the critical mass. Our goal is to give results towards resolving this conjecture, which will be detailed in the next subsection.

II.2.1.3. Main results. In this subsection, we give rigorous statements about our results, which we believe to be the stepping stones for further investigations in the future, both in the theory of viscosity solutions and in the theory of C-F.

First and foremost, we need to understand the existence and uniqueness of viscosity solutions for equation (II.2.1.1).

THEOREM II.2.1.2. *Assume that* $0 < m \leqslant 1$. *Assume further that* F_0 *is Lipschitz, sublinear, and* $0 \leqslant F_0(x) \leqslant mx$. *Then,* (II.2.1.1) *has a unique Lipschitz, sublinear solution* F.

The proof of this theorem is given in Section II.2.2. Theorem II.2.1.2 gives us a simple but important implication about C-F.

COROLLARY II.2.1.3. *Assume that* $m_1(0) = m \in (0, 1]$. *Then, equation* (II.1.1.1) *has at most one mass-conserving solution.*

We believe that the uniqueness result of Corollary II.2.1.3 is new in the literature although existence results of mass-conserving solutions for (II.1.1.1) for the whole range of $m_1(0) \in (0, 1]$ are still not yet available. In a recent important work, Laurençot [Lau19a] showed existence and uniqueness of mass-conserving solutions to (II.1.1.1) under some additional moment conditions for $0 < m_1(0) < \frac{1}{4\log 2}$. In Theorem II.2.1.8 below, we obtain existence (and of course uniqueness) of mass-conserving weak solutions in the measure sense to (II.1.1.1) in case that $0 < m_1(0) < \frac{1}{2}$, and $c(\cdot, 0)$ has bounded second moment.

We note that, in general, if the viscosity solution to the Hamilton-Jacobi equation forms shocks, one cannot have a solution of C-F that conserves mass anymore. This is because if there were a solution of C-F that conserves mass, its Bernstein transform would need to solve the Hamilton-Jacobi equation and at the same time would need to be smooth. This cannot be the case if there were shocks.

It is, therefore, of our interest to study the regularity of the viscosity solutions of the equation (II.2.1.1). Moreover, regularity results in the theory of viscosity solutions are important in their own rights.

THEOREM II.2.1.4. *Suppose $m > 1$. Assume that F_0 is smooth, sublinear, and $0 \leqslant F_0(x) \leqslant mx$. Then equation (II.2.1.1) does NOT admit a solution $F \in C^1([0,\infty)^2)$ which is sublinear in x.*

The proof of this theorem is given in Subsection II.2.3.1. Based on our discussion above, Theorem II.2.1.4 implies immediately the following consequence.

COROLLARY II.2.1.5. *Assume that $m_1(0) = m > 1$. Then, there is no mass-conserving solution to equation (II.1.1.1).*

A version of Corollary II.2.1.5 already appeared in [BLL19]. We here obtain non-existence of mass-conserving solutions under the minimal assumption, that is, $m_1(0) > 1$. We do not need to assume anything else about other moments. In particular, we do not need to impose that the zeroth moment, number of clusters, is finite as in [BLL19]. It is also worth noting that Corollaries II.2.1.3 and II.2.1.5 hold true for mass-conserving weak solutions in the measure sense to (II.1.1.1) as well.

To study regularity of F for $0 < m \leqslant 1$, we impose more conditions on F_0 as following. Assume that there exist $\beta \in (0,1)$ and $C > 0$ such that

$$0 \leqslant F_0'(x) \leqslant m \quad \text{and} \quad F_0'(0) = m\,, \tag{A1}$$

$$-C \leqslant F_0''(x) \leqslant 0\,, \tag{A2}$$

$$-\frac{m}{e} \leqslant xF_0''(x) \leqslant 0 \quad \text{and} \quad \|xF_0''\|_{C^{0,\beta}([0,\infty))} \leqslant C\,. \tag{A3}$$

The above assumptions hold true when F_0 is the Bernstein transform of $c_0 = c(\cdot,0)$, where c_0 has $m_1(0) = m$ and also bounded second moment, that is,

$$m_2(0) = \int_0^\infty s^2 c(s,0)\,ds \leqslant C\,.$$

Indeed,

$$0 \leqslant F_0'(x) = \int_0^\infty s e^{-xs} c(s,0)\,ds \leqslant m\,,$$

and $F_0'(0) = m$. For second derivative, one has

$$-C \leqslant F_0''(x) = -\int_0^\infty s^2 e^{-xs} c(s,0)\, ds \leqslant 0\,,$$

and

$$xF_0''(x) = -\int_0^\infty s^2 x e^{-xs} c(s,0)\, ds = -\int_0^\infty (sxe^{-xs}) s c(s,0)\, ds \geqslant -\frac{m}{e}\,.$$

We use the fact that $re^{-r} \leqslant e^{-1}$ for $r \geq 0$ in the above. Besides, for $x, y \in [0,\infty)$,

$$\begin{aligned} |xF_0''(x) - yF_0''(y)| &\leqslant \int_0^\infty |xe^{-xs} - ye^{-ys}| s^2 c(s,0)\, ds \\ &\leqslant \int_0^\infty |x-y| s^2 c(s,0)\, ds \leqslant C|x-y|. \end{aligned}$$

In the second inequality above, we use the point that $|xe^{-xs} - ye^{-ys}| \leqslant |x-y|$, which can be derived by the usual mean value theorem and

$$\left|(ze^{-zs})'\right| = \left|e^{-zs} - zse^{-zs}\right| \leqslant 1 \qquad \text{for all } z \geqslant 0.$$

We first show that F is always concave in x provided that (A1)–(A2) hold and $0 < m \leqslant 1$.

LEMMA II.2.1.6. *Assume* (A1)–(A2)*, and* $0 < m \leqslant 1$*. Assume further that* F_0 *is sublinear, and* $0 \leqslant F_0(x) \leqslant mx$*. Then, the sublinear solution* F *to the equation* (II.2.1.1) *is concave in* x *for each* $t \geqslant 0$*.*

The concavity of F in the above lemma is rather standard as the Hamiltonian is convex (in fact quadratic) in p. Of course, we need to be careful with the singularity of H in x at $x = 0$, but otherwise, the arguments in the proof of Lemma II.2.1.6 are quite classical. Next, we show that in a smaller range of m ($0 < m < \frac{1}{2}$), $F \in C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$ under assumptions (A1)–(A3). It is worth noting that we do not need to put any assumption on third or higher derivatives of F_0.

THEOREM II.2.1.7. *Assume* (A1)–(A3)*, and* $0 < m < \frac{1}{2}$*. Assume further that* F_0 *is bounded, and* $0 \leqslant F_0(x) \leqslant mx$*. Then the sublinear solution* F *to the equation* (II.2.1.1) *is in* $C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$*. Moreover,* F *satisfies that, for* $(x,t) \in (0,\infty)^2$*,*

$$0 \leqslant \partial_x F(x,t) \leqslant m \quad \text{ and } \quad -1 \leqslant x\partial_x^2 F(x,t) \leqslant 0.$$

To the best of our knowledge, the regularity result in Theorem II.2.1.7 is new in the literature. The proofs of Lemma II.2.1.6 and Theorem II.2.1.7 are given in Subsection II.2.3.2. Next is our existence result for C-F when $0 < m < \frac{1}{2}$.

THEOREM II.2.1.8. *Assume that* F_0 *is the Bernstein transform of* $c_0 = c(\cdot, 0)$*, where* c_0 *has* $m_1(0) = m \in (0, \frac{1}{2})$ *and also bounded zeroth and second moments, that is,*

$$m_0(0) = \int_0^\infty c(s,0)\, ds \leqslant C \quad \textit{and} \quad m_2(0) = \int_0^\infty s^2 c(s,0)\, ds \leqslant C\,.$$

Then (II.1.1.1) *has a mass-conserving weak solution in the measure sense.*

Of course, this mass-conserving weak solution in the measure sense is unique thanks to Corollary II.2.1.3. The range we get here for $0 < m_1(0) < \frac{1}{2}$ is an improvement to the previous range of $0 < m_1(0) < \frac{1}{4\log 2}$ obtained in [Lau19a]. The proof of Theorem II.2.1.8 is given in Subsection II.2.3.2.3. Basically, under the assumptions of Theorem II.2.1.8, we first need to show that $F \in C^\infty((0,\infty)^2)$ in Proposition II.2.3.9. Then, we deduce that $(-1)^{n+1}\partial_x^n F \geqslant 0$ for all $n \in \mathbb{N}$ in Proposition II.2.3.10. These highly nontrivial regularity results of F, together with a characterization of Bernstein functions (see Subsection II.1.2.1 of Chapter 1), allow us to obtain Theorem II.2.1.8.

We then obtain the following large time behavior result for F in case $0 < m < 1$. Here, we do not need assumption (A3).

THEOREM II.2.1.9. *Assume* (A1)–(A2)*. Let* $0 < m < 1$*,* F_0 *be sublinear, and* $0 \leqslant F_0(x) \leqslant mx$*. Let* F *be the Lipschitz, sublinear solution to equation* (II.2.1.1)*. Then*

$$\lim_{t\to\infty} F(x,t) = mx \tag{II.2.1.2}$$

locally uniformly on $[0,\infty)$*.*

REMARK II.2.1.10. It is worth noting that (II.2.5.2) is only useful for $0 < m < 1$, and is meaningless when $m = 1$. Large time behavior of F in case $m = 1$ has been studied recently by Mitake, Tran and Van [MTV21].

Heuristically, Theorem II.2.1.9 implies that as $t \to \infty$, all the solutions (mass-conserving or not) will turn to dusts (particles of size zero) if their initial total mass is less than 1. To see this, we note that, if $F_\infty(x) = \lim_{t\to\infty} F(x,t)$ is a Bernstein transform, then for some measure μ_∞,

$$F_\infty(x) = \int_0^\infty (1 - e^{-sx})\, \mu_\infty(ds) = mx\,.$$

Differentiating in x, it is necessary that

$$\int_0^\infty s e^{-sx}\, \mu_\infty(ds) = m\,,$$

which implies $s\mu_\infty(ds) = m\delta_0(ds)$.

To avoid any confusion, we conclude the introduction by emphasizing the following points.

- While the viscosity solution to the Hamilton-Jacobi equation (II.2.1.1) itself does not correspond to any extension of weak solutions to the C-F, if the viscosity solution F is smooth (i.e., a smooth classical solution) and $(-1)^{n+1}\partial_x^n F \geqslant 0$ in $(0,\infty)^2$ for all $n \in \mathbb{N}$, it would correspond to a mass-conserving weak solution in the measure sense to the C-F. Therefore, regularity of the viscosity solution will imply whether one could have a mass-conserving weak solution in the measure sense to the C-F or not. This is, obviously, an extremely hard and central issue in the theory of viscosity solutions.
- Here, we achieve uniqueness of mass-conserving weak solutions to the C-F for $0 < m_1(0) \leqslant 1$. We show existence of such mass-conserving weak solutions for $0 < m_1(0) < \frac{1}{2}$, and of course, the range $\frac{1}{2} \leqslant m_1(0) \leqslant 1$ is still open.
- To obtain a classical mass-conserving solution for equation (II.1.1.1) in case $0 < m_1(0) < \frac{1}{2}$, one needs to show that the mass-conserving weak solution in the measure sense actually admits a density, which requires more properties from the corresponding Bernstein function. This has been done by Degond, Liu and Pego [DLP17] in a different setting, but remains a hard problem here and will be addressed in future works.

II.2.2. Wellposedness of (II.2.1.1) in case $m \in (0,1]$

We first prove the existence and uniqueness of viscosity solutions to (II.2.1.1). In this section, we always assume that conditions of Theorem II.2.1.2 are in force.

II.2.2.1. Existence of viscosity solutions to (II.2.1.1). We search for sublinear solutions to (II.2.1.1) which satisfy (II.2.1.1b), that is,

$$0 \leqslant F(x,t) \leqslant mx \quad \text{for all } (x,t) \in [0,\infty)^2 .$$

Since (II.2.1.1) is singular at $x = 0$, we cut off its singularity by introducing a sequence of function $\{\phi_n\}$ where

$$\phi_n(x) = \max\left\{\frac{1}{n}, x\right\} \quad \text{for all } x \in [0,\infty) .$$

By the classical theory of viscosity solutions, we have that for each $n \in \mathbb{N}$, the equation

(II.2.2.1)

$$\begin{cases} \partial_t F + \frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{\phi_n(x)} - m = 0 & \text{in } (0,\infty)^2 , \\ F(x,0) = F_0(x) & \text{on } [0,\infty) , \\ F(0,t) = 0 & \text{on } [0,\infty) , \end{cases}$$

has a unique sublinear viscosity solution F^n. In fact, the sublinearity of F^n can be seen through the fact that

$$F_0(x) - Ct \leqslant F^n(x,t) \leqslant F_0(x) + Ct \quad \text{for all } (x,t) \in [0,\infty)^2 ,$$

as $F_0(x) - Ct, F_0(x) + Ct$ are a subsolution and a supersolution to (II.2.2.1), respectively, for some $C > 0$ sufficiently large. To see this, we have

$$C + \frac{1}{2}(\partial_x F_0(x) - m)(\partial_x F_0(x) - m - 1) + \frac{F_0(x) + Ct}{\phi_n(x)} - m \geqslant 0$$

and

$$-C + \frac{1}{2}(\partial_x F_0(x) - m)(\partial_x F_0(x) - m - 1) + \frac{F_0(x) - Ct}{\phi_n} - m \leqslant 0$$

provided that

$$C \geqslant 2m + \sup_{x\in(0,\infty)} |(\partial_x F_0(x) - m)(\partial_x F_0(x) - m - 1)| .$$

Lemma II.2.2.1. *For each $n \in \mathbb{N}$, let F^n be the viscosity solution to equation (II.2.2.1). Then, we have that*

$$F^{n+1} \leqslant F^n \tag{II.2.2.2}$$

for all $n \in \mathbb{N}$.

Proof. To see this, we note that $\phi_n \geqslant \phi_{n+1}$. Therefore

$$\frac{F^n}{\phi_n} \leqslant \frac{F^n}{\phi_{n+1}} ,$$

which implies that F^n is a supersolution to equation (II.2.2.1) with ϕ_{n+1}. Thus, (II.2.2.2) follows. $\square$

Lemma II.2.2.2. *For each $n \in \mathbb{N}$, let F^n be the viscosity solution to equation (II.2.2.1). Then, $\{F^n\}$ is equi-Lipschitz, that is, there exists a constant $C > 0$ so that for every $n \in \mathbb{N}$,*

$$|F^n(x_1,t_1) - F^n(x_0,t_0)| \leqslant C(|t_1 - t_0| + |x_1 - x_0|) , \tag{II.2.2.3}$$

for every $t_0, t_1, x_0, x_1 \in [0,\infty)$.

Proof. We achieve global Lipschitz property in time using the solutions to the approximation problems. We note that equation (II.2.2.1) obeys the classical theory of viscosity solutions so the comparison principle holds.

For each $n \in \mathbb{N}$, we have that $\phi^- \equiv 0$ is a subsolution and $\phi^+ = mx + \frac{1}{n}$ is a supersolution to equation (II.2.2.1). To see the subsolution, we have that

$$\frac{1}{2}m(m+1) - m = \frac{m(m-1)}{2} \leqslant 0 .$$

To see the supersolution, we have that

$$\frac{mx+\frac{1}{n}}{\phi_n(x)} - m = \begin{cases} \frac{1}{nx} & \text{if } x \geqslant \frac{1}{n}\,, \\ nmx + 1 - m & \text{if } x \leqslant \frac{1}{n}\,, \end{cases}$$

which is always nonnegative. On the other hand, as shown just before Lemma II.2.2.1, we also have that $F_0(x) - Ct$ and $F_0(x) + Ct$ are a subsolution and a supersolution to (II.2.2.1), respectively. Therefore, $G^-(x,t) \stackrel{\text{def}}{=} \max\{0, F_0(x) - Ct\}$ is also a subsolution, and $G^+(x,t) \stackrel{\text{def}}{=} \min\{mx + \frac{1}{n}, F_0(x) + Ct\}$ is also a supersolution to (II.2.2.1). And so, by the comparison principle,

$$G^-(x,t) \leqslant F^n(x,t) \leqslant G^+(x,t)\,. \tag{II.2.2.4}$$

Thus, for $t > 0$,

$$|F^n(x,t) - F^n(x,0)| \leqslant Ct\,.$$

By the L^∞-contractive property of solutions to Hamilton-Jacobi equations (which follows from the comparison principle itself), for every $t_0, t_1 \in [0,\infty)$ with $t_1 > t_0$,

(II.2.2.5)

$$\sup_x |F^n(x,t_1) - F^n(x,t_0)| \leqslant \sup_x |F^n(x,t_1 - t_0) - F^n(x,0)| \leqslant C|t_1 - t_0|\,.$$

This is equivalent to the fact that

$$|\partial_t F^n(x,t)| \leqslant C \tag{II.2.2.6}$$

in the viscosity sense. Therefore, rearranging equation (II.2.2.1) and using triangle inequality, estimates (II.2.2.4) and (II.2.2.6), we have

$$|(\partial_x F^n - m)(\partial_x F^n - m - 1)| = 2\left|-\partial_t F^n + m - \frac{F^n}{\phi_n(x)}\right| \leqslant C$$

in the viscosity sense. Therefore, there exists a constant $C > 0$ (independent of $n \in \mathbb{N}$) so that

$$|\partial_x F^n| \leqslant C$$

in the viscosity sense, which is equivalent to

$$|F^n(x_1,t) - F^n(x_0,t)| \leqslant C|x_1 - x_0| \tag{II.2.2.7}$$

for every $x_1, x_0 \in (0,\infty)$. Combining estimates (II.2.2.5) and (II.2.2.7), we get the desired inequality (II.2.2.3). □

LEMMA II.2.2.3. *There exists a function F so that $\{F^n\}$ converges to F locally uniformly on $[0,\infty)^2$, and F is sublinear, uniformly Lipschitz with the same Lipschitz constant as in Lemma II.2.2.2. Furthermore, F is a viscosity solution to equation (II.2.1.1).*

PROOF. The locally uniform convergence follows from Lemmas II.2.2.1 and II.2.2.2. It is clear from the convergence and (II.2.2.4) that F is sublinear, and $0 \leqslant F(x,t) \leqslant mx$ for all $(x,t) \in [0,\infty)^2$. The fact that F is a viscosity solution to (II.2.1.1) follows directly from the definition and the facts that $\{F^n\}$ converges to F locally uniformly and $\{\phi_n\}$ converges to x uniformly. ☐

II.2.2.2. Uniqueness of solutions to (II.2.1.1).

LEMMA II.2.2.4 (Comparison Principle). *Let u be a sublinear viscosity subsolution and v be a sublinear viscosity supersolution to equation* (II.2.1.1)*, respectively. Then $u \leqslant v$.*

PROOF. We have that for every $n \in \mathbb{N}$, u is a subsolution, and $v^n \stackrel{\text{def}}{=} v + \frac{1}{n}$ is a supersolution to equation (II.2.2.1), respectively. The subsolution is clear to see.

To check the supersolution property, we note that, since $m \leqslant 1$,

$$\frac{v + \frac{1}{n}}{\phi_n} - m = \begin{cases} \frac{v}{x} + \frac{1}{nx} - m \geqslant \frac{v}{x} - m, & \text{for } x \geqslant \frac{1}{n}\,, \\ nv + 1 - m \geqslant 0 \geqslant \frac{v}{x} - m, & \text{for } x < \frac{1}{n}\,. \end{cases}$$

Therefore,

$$\begin{aligned} &\partial_t v^n + \frac{1}{2}(\partial_x v^n - m)(\partial_x v^n - m - 1) + \frac{v^n}{\phi_n(x)} - m \\ &\geqslant \partial_t v + \frac{1}{2}(\partial_x v - m)(\partial_x v - m - 1) + \frac{v}{x} - m \geqslant 0 \end{aligned}$$

in the viscosity sense. By the classical theory of viscosity solution applied to equation (II.2.2.1), we imply that

$$u \leqslant v^n\,.$$

But as $v^n \to v$ locally uniformly as $n \to \infty$, we then conclude

$$u \leqslant v\,,$$

as desired. ☐

Let us now give the proof of Theorem II.2.1.2.

Proof of Theorem II.2.1.2. By Lemma II.2.2.3, (II.2.1.1) admits a solution F, which is Lipschitz on $[0,\infty)^2$, and is sublinear in x. Lemma II.2.2.4 then yields the uniqueness of F. ☐

Corollary II.2.1.3 then follows immediately.

Proof of Corollary II.2.1.3. Let c be a mass-conserving solution to (II.1.1.1) with $m = m_1(0) \in (0,1]$. Let F, F_0 be the Bernstein transforms of $c, c_0 = c(\cdot, 0)$, respectively. Then, F is a solution to (II.2.3.1), F is sublinear in x, and $F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2)$. In particular, F is the unique sublinear viscosity solution to (II.2.3.1). This gives the uniqueness of c. ☐

II.2.3. Regularity results

II.2.3.1. Non-existence of C^1 sublinear solutions when $m > 1$. We first show the impossibility of C^1 sublinear solutions when $m > 1$. It is important to note that the sublinear requirement is used crucially here as (II.2.1.1) admits special solutions $\psi_1(x,t) = mx$ and $\psi_2(x,t) = (m-1)x$ for all $(x,t) \in [0,\infty)^2$, which are both linear in x.

Proof of Theorem II.2.1.4. We proceed by contradiction and suppose that such a solution F exists. Then,

$$F(0,t) = 0 \quad \text{and} \quad \partial_t F(0,t) = 0\,.$$

Let $x \to 0^+$ in (II.2.1.1) and use the fact that

$$\partial_x F(0,t) = \lim_{x\to 0^+} \frac{F(x,t) - F(0,t)}{x} = \lim_{x\to 0^+} \frac{F(x,t)}{x}$$

to yield

$$\frac{1}{2}(\partial_x F(0,t) - m)(\partial_x F(0,t) - m - 1) + \partial_x F(0,t) - m = 0\,.$$

Thus, either $\partial_x F(0,t) = m$ or $\partial_x F(0,t) = m-1$. In other words, $\partial_x F(0,t) \geqslant m-1 > 0$. Now, fix $\sigma \in (0, m-1)$. By sublinearity in x of F, for a fixed $t > 0$, there exists $x_t > 0$ such that

$$\varphi(t) \stackrel{\text{def}}{=} \max_{x\in[0,\infty)} (F(x,t) - \sigma x) = F(x_t,t) - \sigma x_t > 0\,.$$

The computations from here to the end of this proof are all justified in the viscosity sense. Observe that, at $x = x_t$, $\partial_x F(x_t,t) = \sigma$ and $F(x_t,t)/x_t > \sigma$. Therefore,

$$\partial_t F(x_t,t) \leqslant -\frac{1}{2}(\sigma - m)(\sigma - m - 1) - (\sigma - m) = -\frac{1}{2}(\sigma - m)(\sigma - m + 1) \stackrel{\text{def}}{=} -c_0\,.$$

Furthermore,

$$\begin{aligned}
\varphi'(t) &= \lim_{s\to 0^+} \frac{\varphi(t) - \varphi(t-s)}{s} \\
&= \lim_{s\to 0^+} \frac{[F(x_t,t) - \sigma x_t] - [F(x_{t-s}, t-s) - \sigma x_{t-s}]}{s} \\
&\leqslant \lim_{s\to 0^+} \frac{[F(x_t,t) - \sigma x_t] - [F(x_t, t-s) - \sigma x_t]}{s} \\
&= \partial_t F(x_t,t) \leqslant -c_0 < 0\,.
\end{aligned}$$

Therefore, there exists $T > 0$ so that $\varphi(T) < 0$, which is a contradiction. $\square$

Proof of Corollary II.2.1.5. Assume by contradiction that there exists a mass-conserving solution c to (II.1.1.1) with $m = m_1(0) > 1$. Let F, F_0 be the Bernstein transforms of $c, c_0 = c(\cdot,0)$, respectively. Then, F is a solution to (II.2.3.1), F is sublinear in x, and $F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2)$. This of course contradicts Theorem II.2.1.4. The proof is complete. $\square$

II.2.3.2. The case $0 < m \leqslant 1$. In the case $0 < m \leqslant 1$, a central topic we set out to study is when is it that classical solutions to the equation (II.2.1.1) exist for all time. This is not a simple task as viscosity solutions to Hamilton-Jacobi equations are Lipschitz, but might not be C^1 in general.

To do this, we study another regularized version of equation (II.2.1.1) by adding a viscosity term and then study the vanishing viscosity limit. Specifically, for $\varepsilon > 0$, we consider

$$\begin{cases} \partial_t F + \frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{x} - m & = \varepsilon a(x)\partial_{xx} F\,, \\ F(x,0) & = F_0(x)\,, \\ F(0,t) & = 0\,. \end{cases} \tag{II.2.3.1}$$

In this section, we use assumptions (A1)–(A3) whenever needed.

We give ourselves some freedom of choices for the nonnegative function $a(x)$. This freedom gives us some flexibility in proving bounds.

II.2.3.2.1. *Concavity of F when $0 < m \leqslant 1$.* In this section, we always assume (A1)–(A2), and F_0 is sublinear, and $0 \leqslant F_0(x) \leqslant mx$. For each $\varepsilon > 0$, let F_1^ε be the classical solution to equation (II.2.3.1) corresponding to $a \equiv 1$. By regularity theory for parabolic equations, $F_1^\varepsilon \in C^\infty((0,\infty)^2) \cap C_1^2([0,\infty) \times (0,\infty))$ (for example, see Ladyženskaja, Solonnikov, Ural'ceva [LSU68], Lieberman [Lie96], Krylov [Kry96]). Here, $C_1^2([0,\infty)\times(0,\infty))$ is the space of functions which are C^2 in x and C^1 in t on $[0,\infty) \times (0,\infty)$.

Lemma II.2.3.1. *Assume (A1)–(A2). Assume further that F_0 is sublinear and $0 \leqslant F_0(x) \leqslant mx$. For each $\varepsilon > 0$, let F_1^ε be the classical solution to equation (II.2.3.1) corresponding to $a \equiv 1$. Then,*

$$0 \leqslant \partial_x F_1^\varepsilon \leqslant m\,. \tag{II.2.3.2}$$

Proof. Firstly, as $0 \leqslant F_1^\varepsilon(x,t) \leqslant mx$ for each $t \geqslant 0$, we imply that

$$0 \leqslant \partial_x F_1^\varepsilon(0,t) \leqslant m\,. \tag{II.2.3.3}$$

Differentiate (II.2.3.1) to get

$$\mathcal{L}^\varepsilon[\partial_x F_1^\varepsilon] + \left(\frac{\partial_x F_1^\varepsilon}{x} - \frac{F_1^\varepsilon}{x^2}\right) = 0\,,$$

where

$$\mathcal{L}^\varepsilon[\phi] \stackrel{\text{def}}{=} \partial_t \phi + \partial_x F_1^\varepsilon \partial_x \phi - (m + \frac{1}{2})\partial_x \phi - \varepsilon \partial_x^2 \phi$$

is a linear parabolic operator.

By Taylor's expansion, for each $(x,t)\in(0,\infty)^2$, there exists $\alpha=\alpha(x,t)\in(0,1)$ so that

$$0=F_1^\varepsilon(0,t)=F_1^\varepsilon(x,t)-x\partial_x F_1^\varepsilon(\alpha x,t)\,.$$

Thus,

$$\mathcal{L}^\varepsilon[\partial_x F_1^\varepsilon]+\frac{\partial_x F_1^\varepsilon(x,t)-\partial_x F_1^\varepsilon(\alpha x,t)}{x}=0\,.$$

We only show here that $\partial_x F_1^\varepsilon \leqslant m$ by the usual maximum principle. The lower bound can be done in a similar manner. Suppose that for some $T>0$, there exists $x_0\geqslant 0$ such that

$$\max_{[0,\infty)\times[0,T]}\partial_x F_1^\varepsilon=\partial_x F_1^\varepsilon(x_0,T)\,.$$

Thanks to (II.2.3.3), we only need to consider the case that $x_0>0$. At this point $\partial_x F_1^\varepsilon(x_0,T)\geqslant \partial_x F_1^\varepsilon(\alpha x_0,T)$, and so $\mathcal{L}^\varepsilon[\partial_x F_1^\varepsilon](x_0,T)\leqslant 0$. By repeating the proof of the maximum principle for a linear parabolic operator, we obtain the desired conclusion that $\partial_x F_1^\varepsilon\leqslant m$. $\square$

Remark II.2.3.2. In the use of the maximum principle, to keep the presentation clean, it is typically the case that one assumes that maximum points of a bounded continuous function ($\partial_x F_1^\varepsilon$ in the above proof) occur. To justify this point rigorously, one can consider maximum of $\partial_x F_1^\varepsilon(x,t)-\delta x$ on $[0,\infty)^2$, for $\delta>0$, and let $\delta\to 0^+$.

Lemma II.2.3.3. *Let F_1^ε be the classical solution to equation* (II.2.3.1) *with $a\equiv 1$. Then,*

$$\partial_x^2 F_1^\varepsilon(0,t)\leqslant 0 \qquad \text{for all } t\geqslant 0\,. \tag{II.2.3.4}$$

Proof. As $F_1^\varepsilon(0,t)=0$ for all $t\geqslant 0$, $\partial_t F_1^\varepsilon(0,t)=0$ and

$$\lim_{x\to 0^+}\frac{F_1^\varepsilon(x,t)}{x}=\partial_x F_1^\varepsilon(0,t)\,.$$

Let $x\to 0^+$ in (II.2.3.1) and use the above to get

$$\frac{1}{2}(\partial_x F_1^\varepsilon(0,t)-m)(\partial_x F_1^\varepsilon(0,x)-m+1)=\varepsilon\partial_x^2 F_1^\varepsilon(0,t)\,, \tag{II.2.3.5}$$

which, together with (II.2.3.2), yields (II.2.3.4). $\square$

We are now ready to prove that F_1^ε is concave in x.

Lemma II.2.3.4. *Assume* (A1)–(A2)*. Assume further that F_0 is sublinear and $0\leqslant F_0(x)\leqslant mx$. For each $\varepsilon>0$, let F_1^ε be the classical solution to equation* (II.2.3.1) *corresponding to $a\equiv 1$. Then, for $(x,t)\in(0,\infty)^2$,*

$$\partial_x^2 F_1^\varepsilon\leqslant 0\,. \tag{II.2.3.6}$$

PROOF. We proceed by the maximum principle. Differentiating (II.2.3.1) twice in x, we get

$$\mathcal{L}^\varepsilon[\partial_x^2 F_1^\varepsilon] + (\partial_x^2 F_1^\varepsilon)^2 + \left(\frac{\partial_x^2 F_1^\varepsilon}{x} + \frac{2(F_1^\varepsilon - x\partial_x F_1^\varepsilon)}{x^3}\right) = 0\,. \tag{II.2.3.7}$$

Recall that

$$\mathcal{L}^\varepsilon[\phi] = \partial_t\phi + \partial_x F_1^\varepsilon \partial_x\phi - (m+\frac{1}{2})\partial_x\phi - \varepsilon\partial_x^2\phi\,.$$

By Taylor's expansion, for each $(x,t)\in(0,\infty)^2$, there exists $\theta=\theta(x,t)\in(0,1)$ so that

$$0 = F_1^\varepsilon(0,t) = F_1^\varepsilon(x,t) - x\partial_x F_1^\varepsilon(x,t) + \frac{x^2}{2}\partial_x^2 F_1^\varepsilon(\theta x,t)\,.$$

This implies

$$\frac{\partial_x^2 F_1^\varepsilon}{x} + \frac{2(F_1^\varepsilon - x\partial_x F_1^\varepsilon)}{x^3} = \frac{\partial_x^2 F_1^\varepsilon(x,t) - \partial_x^2 F_1^\varepsilon(\theta x,t)}{x}\,,$$

which, by plugging into equation (II.2.3.7), gives us

$$\mathcal{L}^\varepsilon[\partial_x^2 F_1^\varepsilon] + (\partial_x^2 F_1^\varepsilon)^2 + \frac{\partial_x^2 F_1^\varepsilon(x,t) - \partial_x^2 F_1^\varepsilon(\theta x,t)}{x} = 0\,.$$

Let us now show that $\partial_x^2 F_1^\varepsilon \leqslant 0$ by the usual maximum principle. Suppose now for some $T>0$, there exists $x_0 \geqslant 0$ so that

$$\max_{[0,\infty)\times[0,T]} \partial_x^2 F_1^\varepsilon = \partial_x^2 F_1^\varepsilon(x_0,T)\,.$$

Thanks to (II.2.3.4), we might assume further that $x_0>0$. By the maximum principle,

$$\mathcal{L}^\varepsilon[\partial_x^2 F_1^\varepsilon](x_0,T) \geqslant 0 \quad \text{and} \quad \partial_x^2 F_1^\varepsilon(x_0,T) - \partial_x^2 F_1^\varepsilon(\theta x_0,T) \geqslant 0\,,$$

which yields

$$(\partial_x^2 F_1^\varepsilon(x_0,T))^2 \leqslant 0 \quad \Rightarrow \quad \partial_x^2 F_1^\varepsilon(x_0,T) = 0\,.$$

This implies $\partial_x^2 F_1^\varepsilon \leqslant 0$, as desired. □

Then, Lemma II.2.1.6 is an immediate consequence of Lemmas II.2.3.1 and II.2.3.4.

II.2.3.2.2. *Regularity of F in case $0<m<\frac{1}{2}$.* Suppose $0<m<\frac{1}{2}$. Here, we always assume (A1)–(A3), and F_0 is sublinear and $0\leqslant F_0(x)\leqslant mx$. Let $a\in C^\infty([0,\infty))$ be a nondecreasing and concave function such that

$$a(x) = \begin{cases} x\,, & x\in[0,1]\,,\\ 2\,, & x\in[3,\infty)\,. \end{cases} \tag{II.2.3.8}$$

For each $\varepsilon>0$, let F_2^ε be the viscosity solution to equation (II.2.3.1) corresponding to the above a. It is worth noting that in this case, (II.2.3.1) is a degenerate parabolic equation, and one needs to be careful with regularity of F_2^ε at $x=0$. Of course, $F_2^\varepsilon\in C^\infty((0,\infty)^2)$, but boundary regularity is not obvious. In the following, we study further properties of F_2^ε by using the specific structure of the equation.

LEMMA II.2.3.5. *For each $\varepsilon > 0$, let F_2^ε be the viscosity solution to equation (II.2.3.1) with a defined as in (II.2.3.8). Then, F_2^ε is concave in x and*

$$0 \leqslant \partial_x F_2^\varepsilon \leqslant m \qquad \text{in } (0,\infty)^2 \,.$$

PROOF. For each $\delta > 0$, consider

(II.2.3.9)
$$\begin{cases} \partial_t F + \frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{x} - m & = (\varepsilon a(x) + \delta)\partial_{xx} F \,, \\ F(x,0) & = F_0(x) \,, \\ F(0,t) & = 0 \,. \end{cases}$$

Let $F_2^{\varepsilon,\delta}$ be the unique solution to the above. Then, $F_2^{\varepsilon,\delta} \in C^\infty((0,\infty)^2) \cap C_1^2([0,\infty) \times (0,\infty))$.

By repeating the proof of Lemma II.2.3.1, we obtain that $0 \leqslant \partial_x F_2^{\varepsilon,\delta} \leqslant m$. In a similar fashion, $\partial_x^2 F_2^{\varepsilon,\delta}(0,t) \leqslant 0$ for all $t \geqslant 0$ by following the proof of (II.2.3.4). Finally, we use the maximum principle to conclude that $F_2^{\varepsilon,\delta}$ is concave in x. Indeed, replicating the proof of Lemma II.2.3.4, we find that for some $T > 0$, there exists $x_0 > 0$ such that

$$\max_{[0,\infty)\times[0,T]} \partial_x^2 F_2^{\varepsilon,\delta} = \partial_x^2 F_2^{\varepsilon,\delta}(x_0, T) \,.$$

The maximum principle then gives us that

$$\left(\partial_x^2 F_2^{\varepsilon,\delta}(x_0,T)\right)^2 \leqslant \varepsilon a''(x_0) \partial_x^2 F_2^{\varepsilon,\delta}(x_0,T) \,.$$

Note that $a''(x_0) \leqslant 0$ as a is chosen to be concave. Therefore, $\partial_x^2 F_2^{\varepsilon,\delta}(x_0,T) \leqslant 0$. Let $\delta \to 0^+$ to get the desired results. $\square$

LEMMA II.2.3.6. *For each $\varepsilon > 0$, let F_2^ε be the viscosity solution to equation (II.2.3.1) with a defined as in (II.2.3.8). Then, $F_2^\varepsilon \in C^1([0,\infty)^2)$ and*

$$\partial_x F_2^\varepsilon(0,t) = m \,.$$

In other words, for $t \geqslant 0$,

(II.2.3.10)
$$\lim_{x\to 0^+} x\partial_x^2 F_2^\varepsilon(x,t) = 0 \,.$$

PROOF. By Lemma II.2.3.5, $x \mapsto \partial_x F_2^\varepsilon(x,t)$ is decreasing in $(0,\infty)$ and $0 \leqslant \partial_x F_2^\varepsilon(x,t) \leqslant m$, and so, $\lim_{x\to 0^+} \partial_x F_2^\varepsilon(x,t)$ exists. By the L'Hopital rule,

$$\partial_x F_2^\varepsilon(0,t) = \lim_{x\to 0^+} \frac{F_2^\varepsilon(x,t) - F_2^\varepsilon(0,t)}{x} = \lim_{x\to 0^+} \partial_x F_2^\varepsilon(x,t) \,,$$

which means that $x \mapsto F_2^\varepsilon(x,t)$ is C^1 on $[0,\infty)$ for each fixed $t \geqslant 0$. Besides, by the results of Daskalopoulos and Hamilton [DH98], Koch [Koc98], Feehan and

Pop [FP13], we yield further that, for each $T > 0$, $F_2^\varepsilon \in \mathcal{C}^{2+\beta}([0,\infty)\times[0,T])$, and

$$\|F_2^\varepsilon\|_{\mathcal{C}^{2+\beta}} \leqslant C\|F_0\|_{\mathcal{C}^{2+\beta}}$$

for some constant $C = C(\varepsilon, T) > 0$. Here,

$$\|f\|_{\mathcal{C}^{2+\beta}} \stackrel{\text{def}}{=} \|f\|_{\mathcal{C}^\beta} + \|\partial_x f\|_{\mathcal{C}^\beta} + \|\partial_t f\|_{\mathcal{C}^\beta} + \|x\partial_x^2 f\|_{\mathcal{C}^\beta},$$

and

$$\|f\|_{\mathcal{C}^\beta} \stackrel{\text{def}}{=} \|f\|_{L^\infty([0,\infty)\times[0,T])} + \sup_{\substack{(x_1,t_1)\neq(x_2,t_2)\\ (x_1,t_1),(x_2,t_2)\in[0,\infty)\times[0,T]}} \frac{|f(x_1,t_1)-f(x_2,t_2)|}{s((x_1,t_1),(x_2,t_2))^\beta}.$$

The distance s is defined as: For $(x_1,t_1),(x_2,t_2) \in [0,\infty)^2$,

$$s((x_1,t_1),(x_2,t_2)) \stackrel{\text{def}}{=} \frac{|x_1-x_2|}{\sqrt{x_1}+\sqrt{x_2}} + \sqrt{|t_1-t_2|}.$$

Let us show now that in fact $\partial_x F_2^\varepsilon(0,t) = m$ for all $t \geqslant 0$. For any $0 < b_1 < b_2$, define by

$$G(x) \stackrel{\text{def}}{=} \int_{b_1}^{b_2} F_2^\varepsilon(x,t)\,dt\,.$$

Integrate (II.2.3.1) with respect to $t \in [b_1,b_2]$ and let $x \to 0^+$ to yield

$$\lim_{x\to 0^+} \varepsilon x\partial_x^2 G(x) = \frac{1}{2}\int_{b_1}^{b_2} (\partial_x F_2^\varepsilon(0,t)-m)(\partial_x F_2^\varepsilon(0,t)-m+1)\,dt \leqslant 0\,.$$

Suppose by contradiction that the right hand side above is negative, which is denoted by $-C < 0$. Then,

$$\lim_{x\to 0^+} x\partial_x^2 G(x) = -\frac{C}{\varepsilon} < 0\,.$$

Thus, by the L'Hopital rule,

$$-\frac{C}{\varepsilon} = \lim_{x\to 0^+} x\partial_x^2 G(x) = \lim_{x\to 0^+} \frac{\partial_x^2 G(x)}{1/x} = \lim_{x\to 0} \frac{\partial_x G(x)}{\log x}\,.$$

However, note that

$$|\partial_x G(x)| = \left|\int_{b_1}^{b_2} \partial_x F_2^\varepsilon(x,t)\,dt\right| \leqslant m(b_2-b_1) = C\,,$$

which implies that

$$\lim_{x\to 0} \frac{\partial_x G(x)}{\log x} = 0\,,$$

which is a contradiction. Thus, we always have $\lim_{\varepsilon\to 0^+} \varepsilon x\partial_x^2 G(x) = 0$ for any $0 < b_1 < b_2$ and, therefore, $\partial_x F_2^\varepsilon(0,t) = m$ for all $t \geqslant 0$. This gives us (II.2.3.10) and also that

$$\lim_{x\to 0^+} \partial_t F_2^\varepsilon(x,t) = 0 = \partial_t F_2^\varepsilon(0,t)\,.$$

The proof is complete. □

LEMMA II.2.3.7. *For each $\varepsilon > 0$, let F_2^ε be the viscosity solution to equation* (II.2.3.1) *with a defined as in* (II.2.3.8). *Then, for $\varepsilon > 0$ sufficiently small,*

$$x\partial_x^2 F_2^\varepsilon \geqslant -1 \quad \text{in } (0,\infty)^2\,. \tag{II.2.3.11}$$

PROOF. We break the proof into a few steps as following.

Step 1. Again, differentiating (II.2.3.1) twice in x, we get

$$\begin{aligned}\Big(\partial_t\partial_x^2 F_2^\varepsilon + \Big[\partial_x F_2^\varepsilon - (m+\frac{1}{2})\Big]\,\partial_x^3 F_2^\varepsilon\Big) + (\partial_x^2 F_2^\varepsilon)^2 + \frac{\partial_x^2 F_2^\varepsilon}{x} - \frac{2\partial_x F_2^\varepsilon}{x^2} + \frac{2F_2^\varepsilon}{x^3}\\ = \varepsilon\Big(a''\partial_x^2 F_2^\varepsilon + 2a'\partial_x^3 F_2^\varepsilon + a\partial_x^4 F_2^\varepsilon\Big)\,.\end{aligned} \tag{II.2.3.12}$$

Let

$$G^\varepsilon \stackrel{\text{def}}{=} x\partial_x^2 F_2^\varepsilon\,.$$

By concavity of F_2^ε in x (Lemma II.2.3.5) and the proof of Lemma II.2.3.6, $G^\varepsilon \in \mathcal{C}^\beta([0,\infty)\times[0,T])$ for each $T > 0$, $G^\varepsilon \leqslant 0$, and $G^\varepsilon(0,t) = 0$ for all $t \geqslant 0$. Besides, by the condition (A3), we have that

$$-\frac{1}{4} \leqslant -\frac{m}{e} \leqslant xF_0''(x) = G^\varepsilon(x,0) \leqslant 0 \qquad \text{for all } x \geqslant 0\,.$$

For $t \geqslant 0$, define by

$$\alpha(t) \stackrel{\text{def}}{=} \inf_{[0,\infty)\times[0,t]} G^\varepsilon.$$

Surely, $\alpha : [0,\infty) \to (-\infty, 0]$ is decreasing and bounded, and $\alpha(0) \in [-\frac{1}{4}, 0]$. We now show that α is continuous. Fix $T > 0$. For $s, t \in [0,T]$, we use the property $G^\varepsilon \in \mathcal{C}^\beta([0,\infty)\times[0,T])$ to see that, for each $x > 0$,

$$|G^\varepsilon(x,s) - G^\varepsilon(x,t)| \leqslant C|s-t|^{\beta/2},$$

for some $C = C(\varepsilon, T) > 0$. Therefore, for $s, t \in [0,T]$,

$$|\alpha(s) - \alpha(t)| \leqslant C|s-t|^{\beta/2}. \tag{II.2.3.13}$$

Thus, α is locally Hölder continuous, and hence, is continuous on $[0,\infty)$. It is of our goal now to show that $\alpha(t) \geqslant -1$ for all $t \in [0,\infty)$.

Step 2. Fix $T > 0$ such that $\alpha(T) < \alpha(0)$. Suppose that there exists $(x_0, t_0) \in (0,\infty)\times(0,T]$ such that

$$\min_{[0,\infty)\times[0,T]} G^\varepsilon(x,t) = G^\varepsilon(x_0,t_0) = \alpha(T) < 0$$

(see Remark II.2.3.8). We then have that, at (x_0, t_0),

$$0 \geqslant \partial_t G^\varepsilon = x_0\partial_t\partial_x^2 F_2^\varepsilon\,,$$

and

$$0 = \partial_x G^\varepsilon = x_0\partial_x^3 F_2^\varepsilon + \partial_x^2 F_2^\varepsilon \iff \partial_x^2 F_2^\varepsilon = -x_0\partial_x^3 F_2^\varepsilon\,, \tag{II.2.3.14}$$

and, therefore,

$$0 \leqslant \partial_x^2 G^\varepsilon \iff x_0^2 \partial_x^4 F_2^\varepsilon \geqslant -2x_0 \partial_x^3 F_2^\varepsilon = 2\partial_x^2 F_2^\varepsilon \,. \tag{II.2.3.15}$$

Multiplying equation (II.2.3.12) by x_0^2 and use estimate (II.2.3.15) to evaluate at (x_0, t_0), we obtain

$$\begin{aligned} &\alpha(T)^2 + \alpha(T)\left(m + \frac{3}{2} - \partial_x F_2^\varepsilon\right) + \frac{2(F_2^\varepsilon - x_0 \partial_x F_2^\varepsilon)}{x_0} \\ &\geqslant \varepsilon\alpha(T)\Big(\frac{2a(x_0)}{x_0} - 2a'(x_0) + a''(x_0)x_0\Big) \geqslant 2\varepsilon\alpha(T)\,. \end{aligned}$$

The last inequality follows since $\alpha(T) \leqslant 0$ and, by the way we choose a,

$$\frac{2a(x_0)}{x_0} - 2a'(x_0) + a''(x_0)x_0 \leqslant \frac{2a(x_0)}{x_0} \leqslant 2\,.$$

Therefore, rearranging terms, we have

$$\alpha(T)^2 + A\alpha(T) + B \geqslant 0\,,$$

where

$$A = m + \frac{3}{2} - 2\varepsilon - \partial_x F_2^\varepsilon(x_0, t_0)\,,$$

and

$$B = \frac{2(F_2^\varepsilon(x_0,t_0) - x_0\partial_x F_2^\varepsilon(x_0,t_0))}{x_0}\,.$$

We have that, since $0 \leqslant \partial_x F_2^\varepsilon \leqslant m$ and F_2^ε is concave in x, for $\kappa = m - \partial_x F_2^\varepsilon(x_0, t_0)$,

$$0 \leqslant \kappa \leqslant m \quad \text{and} \quad 0 \leqslant B \leqslant 2\kappa\,.$$

Therefore,

$$\frac{3}{2} + \kappa - 2\varepsilon = A \leqslant m + \frac{3}{2} - 2\varepsilon < 2\,,$$

and

$$0 \leqslant B \leqslant 2\kappa \leqslant 2m\,.$$

As $0 < m < \frac{1}{2}$, obviously $0 < \kappa \leqslant m < \frac{1}{2}$. For $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} A^2 - 4B &\geqslant \left(\frac{3}{2} + \kappa - 2\varepsilon\right)^2 - 8\kappa \geqslant \frac{9}{4} + \kappa^2 - 5\kappa - 8\varepsilon \\ &= \left(\frac{1}{2} - \kappa\right)\left(\frac{9}{2} - \kappa\right) - 8\varepsilon \\ &\geqslant \left(\frac{1}{2} - \kappa\right)^2 + 4\left(\frac{1}{2} - m\right) - 8\varepsilon > \left(\frac{1}{2} - \kappa\right)^2 > 0\,. \end{aligned}$$

From the quadratic formula and the above estimates, we find that either

$$\alpha(T) \leqslant \frac{-A - \sqrt{A^2 - 4B}}{2}\,,$$

or

$$\alpha(T) \geqslant \frac{-A+\sqrt{A^2-4B}}{2}\,.$$

It is worth noting that, for $\varepsilon>0$ sufficiently small,

$$\frac{-A-\sqrt{A^2-4B}}{2} \leqslant \frac{-A-\left(\frac{1}{2}-\kappa\right)}{2} = -1+\varepsilon \leqslant -\frac{3}{4}-\frac{m}{2}\,,$$

and

$$\frac{-A+\sqrt{A^2-4B}}{2} \geqslant \frac{-A+\left(\frac{1}{2}-\kappa\right)}{2} = -\frac{1}{2}-\kappa+\varepsilon \geqslant -\frac{1}{2}-m\,.$$

We then deduce that, for each $T\geqslant 0$, either

$$\alpha(T) \leqslant -\frac{3}{4}-\frac{m}{2}\,, \tag{II.2.3.16}$$

or

$$\alpha(T) \geqslant -\frac{1}{2}-m\,. \tag{II.2.3.17}$$

Surely, $-\frac{3}{4}-\frac{m}{2}<-\frac{1}{2}-m$ and there is a gap of size $\frac{1-2m}{4}$ between these two numbers.

Step 3. We show that, for small enough $\varepsilon>0$, only (II.2.3.17) holds for all $T\geqslant 0$. Assume by contradiction that this is not the case, then there exists $T>0$ such that (II.2.3.16) holds, that is,

$$\alpha(T) \leqslant -\frac{3}{4}-\frac{m}{2} < -\frac{1}{2}-m < \alpha(0)\,,$$

By the continuity of α, there exists $T^\varepsilon\in(0,T)$ so that

$$-\frac{3}{4}-\frac{m}{2} < \alpha(T^\varepsilon) = \min_{[0,T^\varepsilon]}\alpha < -\frac{1}{2}-m\,,$$

which is a contradiction with the conclusion of Step 2 above.

Thus, for small enough $\varepsilon>0$,

$$x\partial_x^2 F_2^\varepsilon(x,t) \geqslant -\frac{1}{2}-m > -1$$

for every $(x,t)\in(0,\infty)^2$, as desired. □

Remark II.2.3.8. In the use of the maximum principle in the above proof, to keep the presentation clean, we assume that minimum points of G^ε, which is continuous and bounded, exist on $[0,\infty)\times[0,T]$ for $T>0$. To justify this point rigorously, one can consider minimum of $G^\varepsilon(x,t)+\delta x$, for $\delta>0$, and let $\delta\to 0^+$. Let us supply the details here.

Pick $T>0$ such that

$$\alpha(T) = \min_{[0,T]}\alpha < \alpha(0).$$

For each $k \in \mathbb{N}$ sufficiently large, we choose $\delta_k \in (0, \frac{1}{k})$ sufficiently small such that

$$\alpha(T) \leqslant \min_{[0,\infty)\times[0,T]} (G^\varepsilon(x,t) + \delta_k x) = G^\varepsilon(x_k, t_k) + \delta_k x_k \leqslant \alpha(T) + \frac{1}{k} < \alpha(0),$$

for some $(x_k, t_k) \in (0, \infty) \times (0, T]$. In particular, $\delta_k x_k \leqslant \frac{1}{k}$. Let

$$\alpha_k = G^\varepsilon(x_k, t_k) \in \left(\alpha(T), \alpha(T) + \frac{1}{k}\right).$$

We use the maximum principle at (x_k, t_k) and perform careful computations to deduce that

$$\alpha_k^2 + \alpha_k \left(m + \frac{3}{2} - \partial_x F_2^\varepsilon\right) + \frac{2(F_2^\varepsilon - x_k \partial_x F_2^\varepsilon)}{x_k} + \delta_k x_k \left(m + \frac{1}{2} + 2\varepsilon a'(x_k) - \partial_x F_2^\varepsilon\right)$$
$$\geqslant \varepsilon\alpha_k \left(\frac{2a(x_k)}{x_k} - 2a'(x_k) + a''(x_k)x_k\right) \geqslant 2\varepsilon\alpha_k\,.$$

Let $k \to \infty$ and argue in a similar way as in Step 2 of the above proof to yield that either

$$\alpha(T) \leqslant -\frac{3}{4} - \frac{m}{2},$$

or

$$\alpha(T) \geqslant -\frac{1}{2} - m,$$

from which the proof follows. As this is of course tedious and distracting, we intentionally avoid putting it in the above already technical proof.

We are now ready to prove one of our main regularity results that $F \in C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$ when $0 < m < \frac{1}{2}$.

Proof of Theorem II.2.1.7. From Lemma II.2.3.5, Lemma II.2.3.6 and Lemma II.2.3.7, we have that $|\partial_x F_2^\varepsilon| \leqslant m$, $|x\partial_x^2 F_2^\varepsilon| \leqslant 1$ and $|\partial_t F_2^\varepsilon| \leqslant C$. Thus, by the Arzelà-Ascoli theorem, there exists F in $C([0,\infty)^2)$ and a subsequence $\{\varepsilon_i\} \to 0$ so that, locally uniformly

$$\lim_{i\to\infty} F_2^{\varepsilon_i} = F\,.$$

By stability of viscosity solutions, F solves equation (II.2.1.1).

Now, fix $x_0 > 0$. For $x > x_0$, by Lemmas II.2.3.5 and II.2.3.7,

$$-\frac{1}{x_0} \leqslant \partial_x^2 F_2^\varepsilon(x) \leqslant 0\,.$$

Letting $x_1, x_2 > x_0$, we have

$$\left|\frac{\partial_x F_2^\varepsilon(x_1,t) - \partial_x F_2^\varepsilon(x_2,t)}{x_1 - x_2}\right| = \left|\frac{\int_{x_2}^{x_1} \partial_x^2 F_2^\varepsilon(x,t)\,dx}{x_1 - x_2}\right| \leqslant \frac{1}{x_0}\,. \tag{II.2.3.18}$$

Thus, there exist constants $C > 0$ and $z_0 > 0$, such that for $x > x_0$ and $0 < z < z_0$, we can uniformly bound the double difference quotient

$$\left|\frac{F_2^\varepsilon(x+2z,t) - 2F_2^\varepsilon(x+z,t) + F_2^\varepsilon(x,t)}{z^2}\right| \leqslant \frac{C}{x_0}\,.$$

Letting ε to 0, we get

$$\left|\frac{F(x+2z,t) - 2F(x+z,t) + F(x,t)}{z^2}\right| \leqslant \frac{C}{x_0}\,.$$

This implies F is $C^{1,1}$ in x on $[x_0,\infty)\times(0,\infty)$ for all $x_0 > 0$, which yields that F is locally $C^{1,1}$ in x in $(0,\infty)^2$. It is clear then that F is concave and F inherits estimate (II.2.3.11) from F_2^ε, that is,

$$-1 \leqslant x\partial_x^2 F(x,t) \leqslant 0 \quad \text{ for all } (x,t) \in (0,\infty)^2.$$

On the other hand, differentiating equation (II.2.1.1) in x, we have

$$\partial_t U + \partial_x U\left(U - m - \frac{1}{2}\right) + \frac{U}{x} - \frac{F}{x^2} = 0$$

in the viscosity sense, where $U = \partial_x F$.

Now, letting $x > x_0$, by the obtained estimates on F,

$$0 \leqslant U(x,t) \leqslant m, \quad 0 \leqslant \frac{F(x,t)}{x} \leqslant m \quad \text{ and } \quad -\frac{1}{x_0} \leqslant \partial_x U(x,t) \leqslant 0\,.$$

Therefore, there exists $C = C(x_0)$ such that for $x > x_0$,

$$|\partial_t U(x,t)| = |\partial_{tx}^2 F(x,t)| \leqslant C$$

in the viscosity sense. In a similar way, differentiate the equation with respect to t to deduce that for $x > x_0$ and $t > 0$, there exists $C = C(x_0)$ such that

$$|\partial_t^2 F(x,t)| \leqslant C\,.$$

Therefore, $F \in C^{1,1}((0,\infty)^2)$, and F is concave in x. A similar argument (but easier) as that in the proof of Lemma II.2.3.6 shows that $F \in C^1([0,\infty)\times(0,\infty))$. □

II.2.3.2.3. *Existence of solutions to equation* (II.1.1.1) *for* $0 < m_1(0) < \frac{1}{2}$. We now prove the existence of mass-conserving weak solutions in the measure sense to equation (II.1.1.1) when $0 < m = m_1(0) < \frac{1}{2}$. Therefore, in this subsection, we will always assume F_0 is the Bernstein transform of $c_0 = c(\cdot,0)$, where c_0 has $m_1(0) = m \in (0,\frac{1}{2})$ and also bounded second moment, that is,

$$m_2(0) = \int_0^\infty s^2 c(s,0)\,ds \leqslant C\,.$$

Our goal is to show, via a combination of the maximum principle and localizations around the characteristics (see Evans [Eva10, Chapter 3]), that F is a

Bernstein function and, therefore, has a representation as a Bernstein transform of a measure.

By Theorem II.2.1.7, we already have that $F \in C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$. Let us now use this result to yield further that $F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2)$.

Proposition II.2.3.9. *Assume all the assumptions in Theorem II.2.1.8. Then $F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2)$.*

Proof. We proceed by using characteristics and earlier results. Denote by $X(x,t)$ the characteristic starting from x, that is, $X(x,0) = x$. Set $P(x,t) = \partial_x F(X(x,t),t))$, and $Z(t) = F(X(x,t),t)$ for all $t \geqslant 0$. When there is no confusion, we just write $X(t), P(t), Z(t)$ instead of $X(x,t), P(x,t), Z(x,t)$, respectively. Then, $X(0) = x$, $P(0) = \partial_x F_0(x)$, $Z(0) = F_0(x)$. We have the following Hamiltonian system

$$\begin{cases} \dot X = \partial_p H(P(t), Z(t), X(t)) = P(t) - \left(m + \frac{1}{2}\right) , \\ \dot P = -\partial_x H - (\partial_z H) P = \frac{Z(t)}{X(t)^2} - \frac{P(t)}{X(t)} , \\ \dot Z = P \cdot \partial_p H - H = \frac{P(t)^2}{2} - \frac{Z(t)}{X(t)} + \frac{m(1-m)}{2} . \end{cases}$$

Note first that $F \in C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$, and also $0 \leqslant \partial_x F \leqslant m$ thanks to Theorem II.2.1.7. Therefore,

$$-1 \leqslant -\left(m + \frac{1}{2}\right) \leqslant \dot X \leqslant -\frac{1}{2} . \tag{II.2.3.19}$$

Besides, the concavity of F in x yields further that

$$\dot P = \frac{Z(t)}{X(t)^2} - \frac{P(t)}{X(t)} = \frac{1}{X(t)} \left(\frac{F(X(t),t)}{X(t)} - \partial_x F(X(t),t) \right) \geqslant 0 .$$

Let us now show that $\{X(x,\cdot)\}_{x \in (0,\infty)}$ are well-ordered in $(0,\infty)^2$, and none of these two intersect. Assume otherwise that $X(x,t) = X(y,t) > 0$ for some $x \neq y$ and $t > 0$. As $F \in C^{1,1}((0,\infty)^2) \cap C^1([0,\infty) \times (0,\infty))$, $\partial_x F(X(x,t),t)$ is uniquely defined, and therefore,

$$P(x,t) = P(y,t) = \partial_x F(X(x,t),t) \quad \text{and} \quad Z(x,t) = Z(y,t) = F(X(x,t),t) .$$

Hence, $(X,P,Z)(x,t) = (X,P,Z)(y,t)$, and this contradicts the uniqueness of solutions to the Hamiltonian system on $[0,t]$ as we reverse the time.

Next, for each $t > 0$, let $l(t) > 0$ be such that $X(l(t),t) = 0$. This is possible because of (II.2.3.19). As F_0 is smooth, X, P, Z are smooth in x. Thanks to our Hamiltonian system and the well-ordered of $\{X(x,\cdot)\}_{x \in (0,\infty)}$, the map $x \mapsto X(x,t)$ is a smooth bijection from $(l(t),\infty)$ to $(0,\infty)$. Let $X^{-1}(\cdot,t) : (0,\infty) \to (l(t),\infty)$ be the inverse of $X(\cdot,t)$.

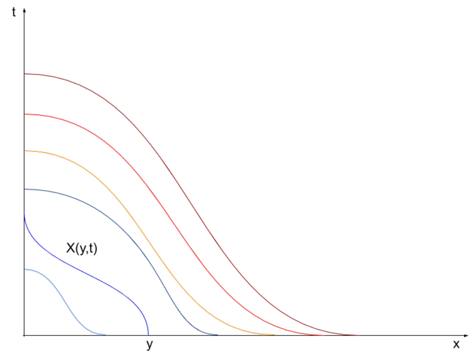

FIGURE II.2.3.1. Characteristics

Let us show further that $X(\cdot,t)$ is a smooth diffeomorphism. It is enough to show that $X(\cdot,t) : (l(t)+n^{-1},n) \to (X(l(t)+n^{-1},t), X(n,t))$ is a smooth diffeomorphism for each $n \in \mathbb{N}$ sufficiently large. Let

$$O = \left\{(X(x,s),s) \,:\, x \in (l(t)+n^{-1},n), s \in [0,t]\right\}.$$

Thanks to Theorem II.2.1.7, there exists $C > 0$ such that

$$-C \leqslant \partial_x^2 F(x,s) \leqslant 0 \qquad \text{in } O$$

in the viscosity sense. We differentiate the first equation in the Hamiltonian system with respect to x and use the fact that $P(x,s) = \partial_x F(X(x,s),s)$ to yield that

$$\partial_x \dot{X}(x,s) = \partial_x P(x,s) = \partial_x^2 F(X(x,s),s) \cdot \partial_x X(x,s) \geqslant -C\partial_x X(x,s).$$

Thus, $\partial_x X(x,s)$ satisfies a differential inequality, and in particular,

$$s \mapsto e^{Cs}\partial_x X(x,s) \qquad \text{is nondecreasing on } [0,t].$$

It is then clear that $\partial_x X(x,s) > 0$ for all $x \in (l(t)+n^{-1},n)$, $s \in [0,t]$. By the inverse function theorem, $X^{-1}(\cdot,t)$ is then smooth, and

$$F(x,t) = Z(X^{-1}(x,t),t)$$

is smooth as Z is also smooth.

Let us finally use the property $\dot{P} \geqslant 0$ to yield that $F \in C^1([0,\infty)^2)$. We only need to show that $\partial_x F$ is continuous at $(0,0)$. For each $\varepsilon > 0$, we are able to find $r > 0$ such that $F_0'(x) \in [m-\varepsilon, m]$ for all $x \in [0,r]$. Let

$$V_r = \{(y,s) \in [0,\infty)^2 \,:\, y = X(x,s) \text{ for some } x \in [0,r] \text{ and } s \geqslant 0\}.$$

Then, as $\dot{P} \geqslant 0$, we see that $\partial_x F(y,s) \in [m-\varepsilon, m]$ for all $(y,s) \in V_r$. The proof is complete. $\square$

It is worth noting that in this problem, for the characteristics, only the condition for $t = 0$ is in use. The boundary condition for $x = 0$, though still satisfied, is not being used (ineffective).

Now that we have $F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2)$, we continue to prove the last requirement to have that F is a Bernstein function.

PROPOSITION II.2.3.10. *Assume all the assumptions in Theorem II.2.1.8. Then,*

$$(-1)^{n+1}\partial_x^n F \geqslant 0 \qquad \text{in } (0,\infty)^2 \text{ for all } n \in \mathbb{N}\,.$$

Of course, we verified the above claim already when $n = 1$. A main difficulty to achieve this result is that $\partial_x^n F$ might be singular at $x = 0$, and thus, we do not have much knowledge on the boundary behavior there. This is also clear in view of the method of characteristics as described above. Here is a way to fix this issue, which is motivated by Lemma II.2.3.6.

LEMMA II.2.3.11. *We have that, for all $t \geqslant 0$,*

$$\lim_{x\to 0^+} x\partial_x^2 F(x,t) = 0\,.$$

PROOF. Let $Q = \partial_x^2 F$. Differentiate (II.2.1.1) with respect to x twice, we get

$$\partial_t Q - \left(m + \frac{1}{2} - \partial_x F\right)\partial_x Q = -Q^2 - \frac{Q}{x} + 2\frac{x\partial_x F - F}{x^3}\,. \tag{II.2.3.20}$$

A very important point here is that (II.2.3.20) has the same characteristics $X(x,t)$ as in Proposition II.2.3.9. Recall that

$$\dot{X} = -\left(m + \frac{1}{2}\right) + \partial_x F(X(t),t)\,,$$

and (II.2.3.19) holds. Let $R(t) = Q(X(t),t)$, then

$$\dot{R} = -R^2 - \frac{R}{X} + 2\frac{XP - Z}{X^3}\,.$$

Since $-1 \leqslant x\partial_x^2 F \leqslant 0$, we infer that $R \leqslant 0$, $1 + RX \geqslant 0$, and

$$\dot{R} = -R^2 - \frac{R}{X} + 2\frac{XP - Z}{X^3} \geqslant 2\frac{XP - Z}{X^3} = \frac{2}{X^2}\left(P - \frac{Z}{X}\right)\,. \tag{II.2.3.21}$$

This differential inequality about R will be used to give us the desired result. Note that $F \in C^1([0,\infty)^2)$, and for each $t \geqslant 0$,

$$\lim_{x\to 0^+}\left(\partial_x F(x,t) - \frac{F(x,t)}{x}\right) = 0\,.$$

So, for fixed $T > 0$, there exists a modulus of continuity $\omega : (0,\infty) \to [0,\infty)$ with $\lim_{r\to 0^+}\omega(r) = 0$ such that for all $r > 0$,

$$\left|\partial_x F(x,t) - \frac{F(x,t)}{x}\right| \leqslant \omega(r) \qquad \text{for all } (x,t) \in (0,r] \times [0,T]\,.$$

Fix $r > 0$ and on each given characteristic $X(x, \cdot)$, which reaches 0 in finite time, take $s_0 \geqslant 0$ such that $0 < X(x, s_0) \leqslant r$. For $s \geqslant s_0$, we use this in (II.2.3.21) to get that

$$\dot{R}(s) \geqslant -\frac{2\omega(r)}{X(s)^2}\,.$$

Integrate this and use (II.2.3.19) to yield, for $t \geqslant s_0$,

$$\begin{aligned} R(t) &\geqslant R(s_0) - 2\omega(r)\int_{s_0}^{t} \frac{1}{X(s)^2}\,ds \geqslant R(s_0) - 2\omega(r)\int_{s_0}^{t} \frac{-2\dot{X}(s)}{X(s)^2}\,ds \\ &= R(s_0) - 4\omega(r)\left(\frac{1}{X(t)} - \frac{1}{X(s_0)}\right). \end{aligned}$$

Thus,

$$X(t)R(t) \geqslant X(t)R(s_0) - 4\omega(r)\,.$$

Besides, $X(t)R(t) \leqslant 0$ thanks to Theorem II.2.1.7. Combine the two inequalities, we get, for $X(t) \leqslant r$ and $t \in [0, T]$,

$$|X(t)R(t)| \leqslant CX(t) + 4\omega(r), \tag{II.2.3.22}$$

where $C = \max_{x\in[0,r]} |F_0''(x)| + \max_{t\in[0,T]} |\partial_x^2 F(r, t)|$. Let $X(t) \to 0^+$ and $r \to 0^+$ in this order in the above to get the conclusion. □

LEMMA II.2.3.12. *Fix $n \in \mathbb{N}$ with $n \geqslant 2$, and $R > 0$. Then, there exists a constant $C = C(n, R) > 0$ such that*

$$\|x^{n-1}\partial_x^n F(x, t)\|_{L^\infty((0,R)^2)} \leqslant C\,. \tag{II.2.3.23}$$

PROOF. The proof is rather tedious with a lot of terms appearing in the differentiations. We prove by induction with respect to $j = n$ in (II.2.3.23). The base case $j = 2$ was already done by Theorem II.2.1.7. Assume that (II.2.3.23) holds true for $j = n - 1 \geqslant 2$, and we now show that it is also true for $j = n$.

Step 1. Differentiate (II.2.1.1) with respect to x by n times, we get

$$\partial_t\partial_x^n F - \left(m + \frac{1}{2}\right)\partial_x^{n+1} F + \frac{1}{2}\partial_x^n\left((\partial_x F)^2\right) + \partial_x^n\left(\frac{F}{x}\right) = 0\,.$$

Let $Q = \partial_x^n F$. Then

$$\partial_t Q - \left(m + \frac{1}{2} - \partial_x F\right)\partial_x Q = f(x, t)\,, \tag{II.2.3.24}$$

where the source term f is

$$f(x, t) = -n(\partial_x^2 F)Q - \frac{Q}{x} - \frac{1}{2}\sum_{k=2}^{n-2} \frac{n!(\partial_x^{k+1}F)(\partial_x^{n+1-k}F)}{k!(n-k)!} - \sum_{k=0}^{n-1} \frac{(-1)^{n-k}n!(\partial_x^k F)}{k!\; x^{n-k+1}}\,.$$

Recall that (II.2.3.24) has the same characteristics $X(x, t)$ as in Proposition II.2.3.9

$$\dot{X} = -\left(m + \frac{1}{2}\right) + \partial_x F(X(t), t)\,,$$

and (II.2.3.19) holds. Thanks to Lemma II.2.3.11 and (II.2.3.22), for fixed $T>0$, we are able to find a modulus of continuity $\omega:(0,\infty)\to[0,\infty)$ with $\lim_{r\to 0^+}\omega(r)=0$ such that

$$\left|x\partial_x^2 F(x,t)\right|\leqslant\omega(r)\quad\text{ for all }(x,t)\in(0,r]\times[0,T]\,.$$

Let $R(t)=Q(X(t),t)$ and fix $r>0$. As $X(t)$ reaches 0 in finite time, we can pick $s_0\geqslant 0$ to be the smallest constant such that $X(s_0)\leqslant r$. Surely, $s_0=0$ in case $X(0)=x\leqslant r$. Without loss of generality, we assume that for some $t\geqslant s_0$, $X(t)>0$, and

$$M\overset{\text{def}}{=}X(t)^{n-1}|R(t)|=\max_{s\in[s_0,t]}X(s)^{n-1}|R(s)|>0\,. \tag{II.2.3.25}$$

Step 2. It is our goal to bound $X(t)^{n-1}R(t)$ uniformly in x. Again, without loss of generality, we may assume that $R(s)$ does not change sign for $s\in(s_0,t]$ (otherwise, change s_0 to be a bigger constant such that $R(s_0)=0$ and $R(s)$ does not change sign for $s\in(s_0,t]$). Let us note right away that $-\frac{Q}{X}=-\frac{R}{X}$ is a good term and needs not to be controlled. Indeed, if $R>0$ in (s_0,t), then $-\frac{R}{X}\leqslant 0$ there, and so

(II.2.3.26)
$$\begin{aligned}|R(t)|=R(t)&=R(s_0)+\int_{s_0}^t f(X(s),s)\,ds\\&\leqslant R(s_0)+\int_{s_0}^t -n(\partial_x^2F)R\,ds\\&\quad+\int_{s_0}^t\left(-\frac{1}{2}\sum_{k=2}^{n-2}\frac{n!(\partial_x^{k+1}F)(\partial_x^{n+1-k}F)}{k!(n-k)!}-\sum_{k=0}^{n-1}\frac{(-1)^{n-k}n!(\partial_x^kF)}{k!X^{n-k+1}}\right)ds\,.\end{aligned}$$

A similar claim holds in case $R<0$ in (s_0,t). A key point that we need here in order to bound the above complicated sum is that, for $i\geqslant 2$, by (II.2.3.19)

$$\int_{s_0}^t\frac{1}{X(s)^i}\,ds\leqslant\int_{s_0}^t\frac{-2\dot X(s)}{X(s)^i}\,ds\leqslant\frac{2}{i-1}\left(\frac{1}{X(t)^{i-1}}-\frac{1}{r^{i-1}}\right)\,. \tag{II.2.3.27}$$

This, together with the induction hypothesis, gives us that

(II.2.3.28)
$$X(t)^{n-1}\left|\int_{s_0}^t\left(-\frac{1}{2}\sum_{k=2}^{n-2}\frac{n!(\partial_x^{k+1}F)(\partial_x^{n+1-k}F)}{k!(n-k)!}-\sum_{k=0}^{n-1}\frac{(-1)^{n-k}n!(\partial_x^kF)}{k!X^{n-k+1}}\right)ds\right|\leqslant C\,.$$

Let us next bound the remaining term containing R. As $-\omega(r)\leqslant x\partial_x^2F\leqslant 0$ in $(0,r]\times[0,T]$, one has

(II.2.3.29)
$$\begin{aligned}n\left|\int_{s_0}^t(\partial_x^2F)R\,ds\right|&\leqslant\int_{s_0}^t\frac{n\omega(r)M}{X(s)^n}\,ds\leqslant\frac{2n\omega(r)M}{n-1}\left(\frac{1}{X(t)^{n-1}}-\frac{1}{r^{n-1}}\right)\\&\leqslant\frac{3\omega(r)M}{X(t)^{n-1}}\leqslant\frac{M}{2X(t)^{n-1}}\end{aligned}$$

for $r>0$ small enough.

Combining (II.2.3.25), (II.2.3.26), (II.2.3.28) and (II.2.3.29), we deduce that

$$M \leqslant C + \frac{M}{2},$$

which yields that $M \leqslant 2C$. By definition of M, $X(t)$ and $R(t)$, we reach the desired result. □

We are now ready to prove Proposition II.2.3.10 by induction. Our idea here is to use the maximum principle for $x^{k-1}\partial_x^k F$ for $k \geqslant 3$ in the inductive argument. However, as the behavior of $x^{k-1}\partial_x^k F$ is unclear as $x \to 0^+$, we need to use localizations around characteristics to take care of this issue.

Proof of Proposition II.2.3.10. Let us show that $(-1)^{n+1}\partial_x^n F \geqslant 0$ in $(0,\infty)^2$ by induction. By Theorem II.2.1.7, this is true for $n = 2$ already. Assume that this is true for all $n \leqslant k-1$ for some $k \geqslant 3$. We now show that this is true for $n = k$. Let us just deal with the case that k is even as the other case can be done analogously.

Step 1. Differentiate (II.2.1.1) with respect to x by k times, we get

$$\partial_t\partial_x^k F - \left(m + \frac{1}{2}\right)\partial_x^{k+1} F + \frac{1}{2}\partial_x^k\left((\partial_x F)^2\right) + \partial_x^k\left(\frac{F}{x}\right) = 0\,. \tag{II.2.3.30}$$

Let $W(x,t) = x^{k-1}\partial_x^k F$, and we aim at deriving a PDE for W. As always, the last term on the left hand side above is not so easy to deal with. The following is a new insight to handle this term thanks to Lemma II.2.3.12,

$$\begin{aligned} x^{k-1}\partial_x^k\left(\frac{F}{x}\right) &= x^{k-1}\partial_x^k\left(\int_0^1 \partial_x F(rx,t)\,dr\right) \\ &= x^{k-1}\int_0^1 r^k \partial_x^{k+1} F(rx,t)\,dr = \frac{1}{x^2}\int_0^x z^k\partial_x^{k+1}F(z,t)\,dz \\ &= \frac{W(x,t)}{x} - \frac{k}{x^2}\int_0^x W(z,t)\,dz\,. \end{aligned}$$

We used integration by parts in the last equality above. Multiply (II.2.3.30) by x^{k-1} and use the above identity, we arrive at

$$\begin{aligned} &\partial_t W - \left(m + \frac{1}{2} - \partial_x F\right)\left(\partial_x W - (k-1)\frac{W}{x}\right) + \frac{W}{x} - \frac{k}{x^2}\int_0^x W(z,t)\,dz \\ &\qquad = -k(\partial_x^2 F)W - x^{k-1}\sum_{i=2}^{k-2}\frac{k!(\partial_x^{i+1}F)(\partial_x^{k+1-i}F)}{i!(k-i)!}\,. \end{aligned} \tag{II.2.3.31}$$

Again, this equation has the same characteristics $X(x,t)$ as in Proposition II.2.3.9,

$$\dot{X} = -\left(m + \frac{1}{2}\right) + \partial_x F(X(t),t)$$

and (II.2.3.19) holds. This clear localization of characteristics is very important.

Step 2. We now need to show that $W \leqslant 0$ in $(0,\infty)^2$. Assume by contradiction that there exists $(x_0, T) \in (0,\infty)^2$ such that $W(x_0,T) > 0$. Of course, $x_0 = X(z,T)$ for some $z > x_0$.

For the initial condition of W, it is not hard to see that $W(0,0) = 0$ and $W(x,0) \leqslant 0$ for $x \in [0,\infty)$. Choose z_1, z_2 very close to z such that $z_1 < z < z_2$, and define a new initial condition $\widetilde{W}(\cdot,0)$, which is smooth on $[0,\infty)$, such that

$$\begin{cases} \widetilde{W}(x,0) = W(x,0) & \text{for } x \in [z_1,z_2]\,, \\ \widetilde{W}(x,0) \leqslant W(x,0) & \text{for } x \notin [z_1,z_2]\,. \end{cases}$$

Let $\widetilde{W}$ be the solution to (II.2.3.31) corresponding to this new initial condition $\widetilde{W}(\cdot,0)$. Because of the locality of the characteristics, we see that $\widetilde{W}(x_0,T) = W(x_0,T)$. In fact, we can choose $\widetilde{W}(\cdot,0)$ to be as negative as we wish outside of $[z_1,z_2]$. For our purpose, we choose z_1, z_2, and $\widetilde{W}(\cdot,0)$ so that

(II.2.3.32)
$$\widetilde{W}(x,t) < \widetilde{W}(X(z,t),t) \quad \text{for all } x \in \left(0, \frac{x_0}{2}\right] \cup [z+1,\infty),\ t \in [0,T]\,.$$

Now, slightly abusing the notations, let us assume that W satisfies (II.2.3.32) as well (in other words, write W in place of $\widetilde{W}$ for simplicity). For each $t \in [0,T]$, by (II.2.3.32), there exists $x_t \in \left(\frac{x_0}{2}, z+1\right)$ so that

$$\xi(t) \stackrel{\text{def}}{=} \max_{x\in[0,\infty)} W(x,t) = W(x_t,t)\,.$$

We use the maximum principle in (II.2.3.31) to get an estimate for ξ. Notice that, as k is even, $(\partial_x^{i+1}F)(\partial_x^{k+1-i}F) \geqslant 0$ for $2 \leqslant i \leqslant k-2$ always by the induction hypothesis. At (x_t,t), we have $\partial_x W(x_t,t) = 0$, and

$$\frac{1}{x_t}\int_0^{x_t} W(z,t)\,dz \leqslant W(x_t,t)\,.$$

Therefore,

$$\xi'(t) + \frac{\xi(t)}{x_t}\left((k-1)\left(m - \frac{1}{2} - \partial_x F\right) + k x_t \partial_x^2 F(x_t,t)\right) \leqslant 0\,.$$

Note that $x_t \in \left(\frac{x_0}{2}, z+1\right)$, and

$$\left|(k-1)\left(m - \frac{1}{2} - \partial_x F\right) + k x_t \partial_x^2 F(x_t,t)\right| \leqslant 2k\,.$$

As $\xi(0) \leqslant 0$, by the usual differential inequality, we get that $\xi(t) \leqslant 0$ for all $t \in [0,T]$. In particular, $0 \geqslant \xi(T) \geqslant W(x_0,T) > 0$, which is absurd. The proof is complete. $\square$

Proof of Theorem II.2.1.8. The result follows immediately by combining Propositions II.2.3.9, II.2.3.10 and Theorem II.1.2.2. $\square$

II.2.4. Equilibria

In this section, we study the equilibria of equation (II.2.3.1) in the case $0 < m \leqslant 1$. At equilibrium, the equation reads

$$\frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{x} - m = 0\,. \tag{II.2.4.1}$$

Let us emphasize again that we search for Lipschitz, sublinear viscosity solution F which satisfies $0 \leqslant F(x) \leqslant mx$ for $x \in [0, \infty)$.

Lemma II.2.4.1. *Suppose $0 < m < 1$. Let F be a Lipschitz, sublinear viscosity solution to equation (II.3.1.5) which satisfies $0 \leqslant F(x) \leqslant mx$ for $x \in [0, \infty)$. Then, there exists a constant $C > 0$ so that all the local minimums of F belong to $[0, C]$.*

Proof. By contradiction, if there exists a sequence of local minimums $\{x_n\} \to \infty$ of F, then by the supersolution test, we have

$$\frac{1}{2}m(m+1) + \frac{F(x_n)}{x_n} - m \geqslant 0\,.$$

This means, for $n \in \mathbb{N}$,

$$\frac{F(x_n)}{x_n} \geqslant \frac{1}{2}m(1-m) > 0\,,$$

which is a contradiction as $F(x_n)/x_n \to 0$ by the sublinearity assumption. ☐

Proposition II.2.4.2. *Suppose $0 < m < 1$. Then equation (II.3.1.5) has no Lipschitz, sublinear viscosity solution F which satisfies $0 \leqslant F(x) \leqslant mx$ for $x \in [0, \infty)$.*

Proof. Suppose by contradiction that F is a Lipschitz, sublinear solution to equation (II.3.1.5) and $0 \leqslant F(x) \leqslant mx$ for $x \in [0, \infty)$. By Lemma II.2.4.1, there exists a $C > 0$ so that $F(x)$ is monotone on $[C, \infty)$, i.e., for a.e. $x \in [C, \infty)$, either

$$F'(x) \geqslant 0 \quad \text{or} \quad F'(x) \leqslant 0\,.$$

Let us consider two cases in the following.

Case 1. $F'(x) \geqslant 0$ for a.e. $x \geqslant C$. Since $F(x) \leqslant mx$, we have

$$\frac{1}{2}(F'(x) - m)(F'(x) - m - 1) = m - \frac{F(x)}{x} \geqslant 0\,.$$

Thus, either $F'(x) \leqslant m$ or $F'(x) \geqslant m + 1$. We claim that $F'(x) \leqslant m$ for a.e. $x \geqslant C$ by changing C to be a bigger value if needed. Indeed, assume otherwise, that this is not the case. Since $F(x) \leqslant mx$, we cannot have that $F'(x) \geqslant m+1$ for a.e. $x > C$. Then, we can find $x_2 > x_1 > C$ such that $F'(x_1) \leqslant m$, and

$F'(x_2) \geqslant m+1$. Let $\phi(x) = (m+\frac{1}{2})x$ for $x \in [x_1, x_2]$ be a test function, and let $x_3 \in [x_1, x_2]$ be a minimum point of $F - \phi$ on $[x_1, x_2]$. As

$$F'(x_1) \leqslant m < \phi'(x_1) = m + \frac{1}{2} = \phi'(x_2) < m+1 \leqslant F'(x_2)\,,$$

it is clear that $x_3 \neq x_1$ and $x_3 \neq x_2$. In other words, $x_3 \in (x_1, x_2)$, and one is able to use the viscosity supersolution test to yield that

$$0 \leqslant \frac{1}{2}\left(m + \frac{1}{2} - m\right)\left(m + \frac{1}{2} - m - 1\right) + \frac{F(x_3)}{x_3} - m \leqslant -\frac{1}{8}\,,$$

which is absurd. Therefore,

$$0 \leqslant F'(x) \leqslant m \qquad \text{for a.e. } x \geqslant C\,.$$

In particular, for a.e. $x \geqslant C$,

$$\frac{1}{2}(F'(x) - m)(F'(x) - m - 1) \leqslant \frac{1}{2}(0 - m)(0 - m - 1) = \frac{1}{2}m(m+1)\,,$$

which implies

$$\frac{F(x)}{x} \geqslant m - \frac{1}{2}m(m+1) = \frac{1}{2}m(1-m) > 0\,.$$

But this means that F is not sublinear.

Case 2. $F'(x) \leqslant 0$ for a.e. $x \geqslant C$. Then F is decreasing on $[C, \infty)$ and there exists $\alpha \geqslant 0$ such that $\alpha = \lim_{x\to\infty} F(x)$. Consequently,

$$\lim_{x\to\infty}\left(\frac{1}{2}(F'(x) - m)(F'(x) - m - 1) - m\right) = 0\,.$$

On the other hand, as $F \geqslant 0$ always, we can find a sequence $\{y_n\} \to \infty$ such that $F'(y_n) \to 0$. Let $x = y_n$ in the above and let $n \to \infty$ to deduce that

$$0 = \frac{1}{2}(0 - m)(0 - m - 1) - m = \frac{1}{2}m(m-1) < 0\,,$$

which is absurd.

Therefore, in all cases, we are led to contradictions. The proof is complete. □

PROPOSITION II.2.4.3. *Let $m = 1$. Then equation* (II.3.1.5) *admits a Lipschitz, sublinear viscosity solution F which satisfies $0 \leqslant F(x) \leqslant mx$ for $x \in [0, \infty)$.*

PROOF. Let $G = 1 - \partial_x F$. Then the equilibrium equation reads as

$$\frac{1}{2}G(G+1) - \frac{1}{x}\int_0^x G = 0\,. \tag{II.2.4.2}$$

This is the same equation studied in the work of Degond, Liu and Pego [DLP17, Section 5], of which the solution must satisfy the transcendental equation

$$\frac{G(x)}{(1 - G(x))^3} = Cx\,, \tag{II.2.4.3}$$

for some constant $C > 0$. Let us recall a quick proof of (II.2.4.3). Multiply (II.2.4.2) by x, then differentiate the result with respect to x to imply

$$\frac{1}{2}G(G+1) + \frac{1}{2}x(2G\partial_x G + \partial_x G) - G = 0\,,$$

which means that

$$\frac{1}{x} = \frac{3\partial_x G}{1-G} + \frac{\partial_x G}{G}\,.$$

Integrate the above to yield (II.2.4.3). Therefore, we can pick $C = 1$ in (II.2.4.3) and G to be a Bernstein function taking the form

$$G(x) = \int_0^\infty (1 - e^{-sx})\gamma(s)e^{-4s/27}\,ds\,, \tag{II.2.4.4}$$

where

$$\int_0^\infty \gamma(s)e^{-4s/27}\,ds = 1\,.$$

See [DLP17, Section 5] for further details on the derivation of (II.2.4.4). This implies that

$$\partial_x F(x) = 1 - \int_0^\infty (1 - e^{-sx})\gamma(s)e^{-4s/27}\,ds \geqslant 0\,, \tag{II.2.4.5}$$

and that

$$\lim_{x\to\infty} \frac{F(x)}{x} = 0\,.$$

Furthermore, by successive differentiations, we can also see that $\partial_x F$ is completely monotone, that is, $(-1)^{n+1}\partial_x^n F \geqslant 0$ for all $n \in \mathbb{N}$, which means that F is a Bernstein function. ☐

Remark II.2.4.4. From the above proposition, it is actually not hard to see that, for $m = 1$, equation (II.3.1.5) admits a family of Lipschitz, sublinear viscosity solution $\{F_\lambda\}_{\lambda>0}$ which satisfies $0 \leqslant F_\lambda(x) \leqslant x$ for $x \in [0,\infty)$. Indeed, take F as in the above proof, and let

$$F_\lambda(x) = \lambda F\left(\frac{x}{\lambda}\right) \qquad \text{for all } x \in [0,\infty)\,.$$

Then,

$$G_\lambda(x) = 1 - \partial_x F_\lambda(x) = 1 - \partial_x F\left(\frac{x}{\lambda}\right) = G\left(\frac{x}{\lambda}\right)\,,$$

which means that

$$\frac{G_\lambda(x)}{(1-G_\lambda(x))^3} = \frac{x}{\lambda}\,.$$

This implies that (II.2.4.3) is satisfied with $C = \frac{1}{\lambda}$. Hence, F_λ is a solution to (II.3.1.5) for each $\lambda > 0$.

The existence of this family of solutions $\{F_\lambda\}_{\lambda>0}$ to (II.3.1.5) makes the study of large time behavior of the viscosity solution to (II.2.1.1) for $m = 1$ quite difficult.

II.2.5. Large time behavior for $0 < m < 1$

In this section, we study the large time behavior of the viscosity solution to equation (II.2.1.1) for $0 < m < 1$. Our goal is to prove Theorem II.2.1.9.

From Proposition II.2.4.2, one cannot expect a sublinear equilibrium, that is, a Lipschitz sublinear solution to (II.3.1.5). However, it is very interesting that the solution to equation (II.2.1.1) still converges to the linear function mx locally uniformly as $t \to \infty$. This implies that, even if we have a mass-conserving solution at all time, the sizes of particles decrease until they become dust at time infinity.

To prove the theorem, we need the following results.

Lemma II.2.5.1. *Let $\bar{F}$ be a viscosity supersolution to equation* (II.3.1.5) *that satisfies the following*

$$\begin{cases} \bar{F} \text{ is concave}, \\ \liminf_{x\to\infty} \frac{\bar{F}(x)}{x} > 0\,, \\ 0 \leqslant \bar{F}(x) \leqslant mx\,. \end{cases} \tag{II.2.5.1}$$

Then, $\bar{F}(x) = mx$.

Proof. First, observe that $x \mapsto \partial_x \bar{F}(x)$ is decreasing whenever $\partial_x \bar{F}(x)$ is defined. By the requirement that

$$\liminf_{x\to\infty} \frac{\bar{F}(x)}{x} > 0\,,$$

we have that $\partial_x \bar{F}(x) \geqslant 0$. As $\bar{F}$ is differentiable almost everywhere, pick $\{x_n\} \to \infty$ so that F is differentiable at x_n for all $n \in \mathbb{N}$. Define

$$0 < \alpha \overset{\text{def}}{=} \lim_{n\to\infty} \partial_x \bar{F}(x_n) = \lim_{x\to\infty} \frac{\bar{F}(x_n)}{x_n} \leqslant m\,.$$

Thus, letting $x_n \to \infty$ in the equation (II.3.1.5), we get

$$0 \leqslant \frac{1}{2}(\alpha - m)(\alpha - m + 1) \leqslant 0\,.$$

Therefore, it is necessary that $\alpha = m$ and $\bar{F}(x) = mx$ for all $x \in [0, \infty)$. $\square$

We immediately have the following consequence.

Corollary II.2.5.2. *Let $\bar{F}$ be a viscosity solution to equation* (II.3.1.5) *satisfying* (II.2.5.1)*. Then $\bar{F}(x) = mx$ for $x \in [0, \infty)$.*

LEMMA II.2.5.3. *Let F be the Lipschitz, sublinear viscosity solution to equation (II.2.1.1). Then, locally uniformly for $x \in [0, \infty)$,*

$$\liminf_{t\to\infty} F(x,t) \geqslant \frac{1}{4} m(1-m)x \,. \tag{II.2.5.2}$$

PROOF. We construct a sublinear subsolution to the equation (II.2.1.1) so that the inequality (II.2.5.2) holds. Define, for $(x,t) \in [0,\infty)^2$,

$$\varphi(x,t) \stackrel{\text{def}}{=} \min\left\{\frac{1}{4}m(1-m)x, \frac{1}{4}m(1-m)t\right\} .$$

To see that φ is a subsolution to (II.2.1.1), we first note that $\frac{1}{4}m(1-m)x$ is a subsolution. Furthermore,

$$\varphi(x,t) = \begin{cases} \frac{1}{4}m(1-m)x\,, & x < t\,, \\ \frac{1}{4}m(1-m)t\,, & x \geqslant t\,. \end{cases}$$

So, for $x > t$,

$$\begin{aligned} &\partial_t\varphi + \frac{1}{2}(\partial_x\varphi - m)(\partial_x\varphi - m - 1) + \frac{\varphi}{x} - m \\ \leqslant &\frac{1}{4}m(1-m) + \frac{1}{2}m(m-1) + \frac{1}{4}m(1-m) = 0\,. \end{aligned}$$

Since equation (II.2.1.1) has a convex Hamiltonian, minimum of two subsolutions is a subsolution (see Tran [Tra21, Chapter 2] and the references therein). Note that this property is not true for general Hamiltonians.

By the comparison principle, we have that $F \geqslant \varphi$. Letting $t \to \infty$, we obtain (II.2.5.2) locally uniformly for $x \in [0,\infty)$. $\square$

Proof of Theorem II.2.1.9. By Lemma II.2.5.3, locally uniformly for $x \in (0, \infty)$,

$$m \geqslant \liminf_{t\to\infty} \frac{F(x,t)}{x} \geqslant \frac{1}{4}m(1-m) > 0\,.$$

Let

$$G(x) \stackrel{\text{def}}{=} \liminf_{t\to\infty} F(x,t) \quad \text{for all } x \in [0,\infty)\,.$$

This function is well-defined since F is globally Lipschitz on $[0,\infty)^2$ and $0 \leqslant F(x,t) \leqslant mx$. By stability of viscosity solutions, G is a supersolution to equation

$$\frac{1}{2}(\partial_x G - m)(\partial_x G - m - 1) + \frac{G}{x} - m \geqslant 0$$

in $(0,\infty)$. As $x \mapsto F(x,t)$ is concave for every $t \geqslant 0$, G is concave. Moreover, $0 \leqslant G \leqslant mx$ and

$$G(x) \geqslant \frac{1}{4}m(1-m)x \quad \text{for all } x \in [0,\infty)\,.$$

By Lemma II.2.5.1, $G(x) = mx$ for $x \in [0, \infty)$. We use this and the fact that $F(x,t) \leqslant mx$ for all $(x,t) \in [0,\infty)^2$ to conclude that, locally uniformly for $x \in [0, \infty)$,

$$\lim_{t\to\infty} F(x,t) = G(x) = mx\,,$$

as desired. $\square$

CHAPTER 3

Dynamics for multiplicative coagulation kernel singularly perturbed by additive fragmentation kernel

II.3.1. Introduction

The main goal of this chapter is to discuss a problem concerning the dynamics of the solution to discrete coagulation-fragmentation equation with multiplicative coagulation and small additive fragmentation kernels. The problem could be thought of as a pure coagulation equation that is singularly perturbed by small fragmentation. Letting the fragmentation kernel vanish, in the limit, one expects that the solutions tend to the so-called Flory solution of the pure multiplicative coagulation equation, where part of the total mass escapes to infinity. We describe how the lost mass behaves. Our proposed technique is based on the study of a nonlinear backward parabolic equation, resulting from the Laplace transform of the equation, and a detailed study of the tail behavior of the Flory solution of the pure coagulation equation with multiplicative kernel. We formulate a variant of a prediction by Ben-Naim and Kaprisky [BK11] and discuss what has been done and what should be done to prove this prediction.

To begin, we mathematically prescribe the kernels under consideration:

$$a(\hat{s}, s) = \hat{s}s \quad \text{and} \quad b(\hat{s}, s) = b^{\varepsilon}(\hat{s}, s) = \varepsilon(\hat{s} + s)\,, \tag{II.3.1.1}$$

where $0 \leqslant \varepsilon \leqslant 1$. In this chapter, we will restrict ourselves to the discrete equation and leave the study of the continuous equation for future work as, to our knowledge, the problem of well-posedness for the continuous solution is still not investigated. Henceforth, we write $\rho_k(t) = \rho(k,t)$ and in the strong form the discrete equation with multiplicative–additive kernels reads

$$\begin{aligned}\partial_t \rho_k(t) =& \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j(t)\rho_{k-j}(t) - k\rho_k(t)\sum_{j=1}^{\infty} j\,\rho_j(t) \\ &+ \sum_{j=1}^{\infty} \varepsilon(k+j)\rho_{k+j}(t) - \frac{1}{2}\varepsilon k(k-1)\rho_k(t)\,,\end{aligned} \tag{II.3.1.2a}$$

with initial value

$$\rho(0) = \rho_0\,. \tag{II.3.1.2b}$$

When $\varepsilon = 0$, our equation becomes a pure coagulation equation with multiplicative coagulation kernel. This pure coagulation equation exhibits a phenomenon called *gelation*, in which the total mass of the system is not conserved for all time $t > 0$ due to the formation of large particles that escape the scale captured by the model (see [McL62; Ley03] for further discussions). One can think of these particles as those with infinite size. An important feature of the pure multiplicative coagulation equation is that after gelation, it may have two solutions, depending on how one looks at the equation initially. Specifically, as the solution conserves mass m until the gelation time, one can either write the pure coagulation equation as

$$\partial_t \rho_k(t) = \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j(t)\rho_{k-j}(t) - k\rho_k(t)\sum_{j=1}^{\infty} j\,\rho_j(t)\,, \tag{II.3.1.3}$$

or

$$\partial_t \rho_k(t) = \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j(t)\rho_{k-j}(t) - km\rho_k(t)\,. \tag{II.3.1.4}$$

While equation (II.3.1.3) only considers the interactions of finite-mass particles after gelation, equation (II.3.1.4) allows interactions between finite-mass particles and infinite-mass ones. The solution to the former equation is called the Stockmayer solution and latter the Flory solution (see [Ley03] for further discussions). Both exhibit gelation and are the same up to the gelation time T_{gel}.

When $\varepsilon > 0$, the equation enters a regime where fragmentation helps prevent the formation of infinite-size particles. Hence, in this case, we have a unique solution that conserves mass, as proven by da Costa [Cos95] for the discrete equation (see Theorem II.3.1.1 below).

We denote the n^{th} moment of $\rho(t)$ by

$$m_n(t) = \sum_{i\in\mathbb{N}} k^n \rho_k(t)\,.$$

We will *always assume finite first and second moments for initial data*, i.e.,

$$m_1(0) + m_2(0) < \infty\,. \tag{A1}$$

For convenience, when there is no notational confusion, we will write

$$m = m_1(0)\,.$$

Furthermore, by solution, *we always mean mass-conserving solution*, whose existence and uniqueness to (II.3.1.2) has been studied by da Costa [Cos95] using the method of moment bounds. Now, we recast without proof his result in the following theorem.

THEOREM II.3.1.1 ([Cos95]). *Assume* (A1). *There exists a unique admissible solution ρ to equation* (II.3.1.2) *that preserves mass, i.e.,*

$$m_1(t) \overset{\text{def}}{=} \sum_{k=1}^{\infty} k\rho_k(t) = \sum_{k=1}^{\infty} k\rho_k(0) \overset{\text{def}}{=} m_1(0)\,.$$

Furthermore, ρ has finite second moment, i.e.,

$$m_2(t) \overset{\text{def}}{=} \sum_{k=1}^{\infty} k^2\rho_k(t) < \infty\,,$$

for all time $t > 0$.

Due to mass conservation when $\varepsilon > 0$, in the limit as ε tends to 0, one expects that solutions to equation (II.3.1.2) converge to the solution of the Flory solution of the pure multiplicative coagulation equation ($\varepsilon = 0$) (see Proposition II.3.1.4). However, as the total mass is not conserved for all time for the Flory solution, a crucial question one may ask is: "What happens to the lost mass?" Our goal is to provide an answer to this question.

Let ρ^0 be the Flory solution and let the loss of mass to infinity be defined by $g : [0, \infty) \to [0, \infty)$
$$g(t) = m - \sum_{k\in\mathbb{N}} k\rho_k^0(t) \,.$$
For small ε, one expects an emergence of giant clusters with larger scale that contain a significant portion of the total mass. By assuming that the number of particles is of order $O(1/\varepsilon)$ and the mass of the giant particles is of order $O(g(t)/\varepsilon)$, Ben-Naim and Krapivsky [BK11] heuristically described the giant particles coming from (II.3.1.2) by the *stationary* solution to the equation
$$\varepsilon\partial_t G(s,t) = \frac{1}{2}\int_0^s \hat{s}(s-\hat{s})G(\hat{s},t)G(s-\hat{s},t)\,d\hat{s} - s(g(t)-s)G(s,t)+ \\ + \int_s^{g(t)} \hat{s}G(\hat{s},t)\,d\hat{s} - \frac{s^2}{2}G(s,t)\,,$$
where G here is the density of the large cluster after being normalized appropriately. The $g(t)$ upper bound in the last integral follows from the assumption that $g(t)/\varepsilon$ is the size of the largest particle before normalization. The key observation is that in the pure coagulation equation, *gelation* occurs and the lost mass in this case will be approximately the mass for the large particles in (II.3.1.2) after some time $T_{\text{gel}} > 0$. It is worth noting that Ben-Naim and Krapivsky's assumption is motivated by stochastic simulations and, hence, the largest particle assumption is natural.

Our analysis does not assume there is a largest particle and takes the coagulation-fragmentation equation as the starting point (as opposed to stochastic simulations like Ben-Naim and Krapivsky). Therefore, the above equation does not quite capture the large clusters and modifications need to be made. Keeping the same scaling as Ben-Naim and Krapivsky did, we will show that the giant particles are approximated by the solution to the equation
$$0 = \frac{1}{2}\int_0^s \hat{s}(s-\hat{s})G(\hat{s},t)G(s-\hat{s},t)\,d\hat{s} - sg(t)G(s,t)+ \\ + \int_s^{\infty} \hat{s}G(\hat{s},t)\,d\hat{s} - \frac{s^2}{2}G(s,t)\,,$$
whose formula is given by
$$G(s,t) = \frac{e^{-s/g(t)}}{s}\,.$$

This formula could be obtained by a certain scaling limit of the equilibrium of the discrete equation (II.3.1.2), whose formula is given in the following proposition.

Proposition II.3.1.2. *The unique non-zero equilibrium solution of equation* (II.3.1.2) *in discrete space is given by*

$$\bar{\rho}_k = \frac{\varepsilon m^k}{k(m+\varepsilon)^k}\,, \tag{II.3.1.5}$$

where m is the total mass, i.e.,

$$m = m_1 \stackrel{\text{def}}{=} \sum_{k\in\mathbb{N}} k\bar{\rho}_k\,.$$

Remark II.3.1.3. An important feature of the equilibrium solution is that it satisfies the detailed-balance condition

$$a(k,j)\bar{\rho}_k\bar{\rho}_j = b(k,j)\bar{\rho}_{k+j}\,.$$

This tells us that the net rate of reaction for one pair of cluster sizes due to coagulation and fragmentation is zero, which is intuitively clear but may not necessarily be satisfied for all kernels.

The question about long-time behavior of the coagulation-fragmentation equation is a challenge in general and has been studied by researchers for different types of kernels (see, for example, [Car92; FM04; MP04; MP08; FM04; Cañ07; Lau19b; BLL19; MTV21]). However, a lot still remains unknown. For our specific kernels, this question has been partially answered. Denote $\mu_k = k\rho_k(t)$ and $\bar{\mu}_k = k\bar{\rho}_k$, where ρ is the solution of (II.3.1.2). It has been proven by Fournier and Mischler [FM04] that μ converges to $\bar{\mu}$ exponentially fast in L^1 if the initial mass is sufficiently small. Exponential decay to equilibrium for large initial mass remains an open question. We will not discuss this issue here but leave it for future work.

As a first step towards our analysis, we show that as ε tends to zero, the loss of mass to infinity does happen and, thus, the pure coagulation equation is expected to be a good approximation of small scale particles in (II.3.1.2).

Proposition II.3.1.4. *Let ρ^0 be the Flory solution to the pure coagulation equation ($\varepsilon = 0$) and ρ^ε be the solution to equation* (II.3.1.2) *for each $\varepsilon \geqslant 0$. Then, for every $k \in \mathbb{N}$,*

$$\lim_{\varepsilon\to 0} \rho_k^\varepsilon = \rho_k^0$$

locally uniformly in t.

In order to understand the behavior of ρ^ε as $\varepsilon \to 0$, it is crucial to understand the tail behavior of the Flory solution. We stress that, while this is a step to our final goal, it is important in general to understand the tail behavior of Flory solution for general initial data as well. To the best of our knowledge, this has not been studied except for the special case when the initial datum is *monomers*, in which case, one can write down an explicit formula for the Flory equation based on induction [Ley03; McL62], namely

$$\rho_k^0(t) = \frac{k^{k-3}}{(k-1)!} t^{k-1} e^{-kt} \,. \tag{II.3.1.6}$$

From here and using Stirling's approximation formula, we could deduce that

$$\rho_k^0(t) \leqslant \frac{C}{k^{5/2}}$$

for all $t \geqslant 0$. We confirm this bound for general initial data with minimal assumption.

THEOREM II.3.1.5. *Let ρ^0 be the Flory solution to the pure coagulation equation ($\varepsilon = 0$). Suppose ρ_0 satisfies (A1). Then, there exists a constant $C > 0$ such that for every $t > 0$, we have*

$$\rho_k^0(t) \leqslant \frac{C}{k^{5/2}} \,. \tag{II.3.1.7}$$

The proof of this theorem is based on the so-called Bernstein transform of the mass density of Flory solution. We study the regularity of the characteristics of the complex Burgers's equation (obtained by performing the Bernstein transform to equation (II.3.1.4)) on the right half of the complex plane and use the Fourier-Laplace inversion formula to deduce the tail behavior of the Flory solution. This technique is inspired by that of Menon and Pego [MP06]. The difficulty in our case is that we have to deal with the formation of shocks of the Burgers's equation. Hence, results in [MP06] do not readily apply.

Viewing $\{\mu^\varepsilon \stackrel{\text{def}}{=} k\rho_k^\varepsilon\}_{\varepsilon>0}$ as a set of measures and using concentration-compactness lemma [Lio84; Str08b], we deduce that there is a split of the mass into two almost disjoint systems of particles, small particles and giant particles. Because of the splitting of μ^ε, one can see that there is a significant mass that concentrates in the giant particles. A natural question that is equivalent to the prediction by Ben-Naim and Krapivsky is, "What is the correct scaling to study the large scale clusters as ε tends to 0?" The following conjecture, if true, would be an answer to this question.

CONJECTURE II.3.1.6. *Define $c^\varepsilon : [0,\infty)^2 \to \mathbb{R}$*

$$c^\varepsilon(s,t) \stackrel{\text{def}}{=} n^2 \rho^n_{\lfloor s/\varepsilon \rfloor}(t) \,,$$

Then, for every $t > 0$, the following weak-star convergence of finite Radon measures on $[0,\infty)$ holds

$$s c^\varepsilon(s,t)\, ds \stackrel{*}{\rightharpoonup} e^{-s/g(t)}\, ds + (m - g(t))\delta_0 \,, \tag{II.3.1.8}$$

as $\varepsilon \to 0$. Furthermore, the function $\bar{c}(s) = s^{-1}e^{-s/g(t)}$ is the solution to the equation

$$\frac{1}{2}\int_0^s \hat{s}(s-\hat{s})\bar{c}(\hat{s},t)\bar{c}(s-\hat{s},t)\,d\hat{s} - sg(t)\bar{c}(s,t) + \\ + \int_s^\infty \hat{s}\bar{c}(\hat{s},t)\,d\hat{s} - \frac{s^2}{2}\bar{c}(s,t) = 0\,. \tag{II.3.1.9}$$

Heuristically, this conjecture says that the giant particles behave as if they were in equilibrium, approximated by the solution of equation (II.3.1.9), given that the mass "near infinity" is $g(t)$. This means that for each given ε, there is a metastable manifold where the solutions to (II.3.1.2) spend some time nearby before converging to the true equilibirum solution of (II.3.1.2).

We propose a direction to resolve Conjecture II.3.1.6 based on rescaling ρ^ε and applying the Laplace transform. By a bad fortune, when $\varepsilon > 0$, the Laplace transform of the scaled solution to (II.3.1.2) unfortunately gives a nonlinear singular backward parabolic equation (see equation (II.3.4.11)). This equation is far from well understood in any sense. We try to overcome this difficulty by exploiting the tail behavior of the Flory solution so that for ε sufficiently small, we can bound the time derivative of the Laplace transform of the scaled solution uniformly. This allows us the ability to treat the equation as a second order elliptic equation, which is well understood.

Even though at the time this thesis is being written the proposed method still does not yield the desired result, it allows us to come very close to the resolution. One of our future goals is to study more deeply what is achievable by this method. We note that our technique is novel in the sense that it tightly harmonizes the information between the solution of the coagulation-fragmentation equation and that of the transformed equation. Previously, works have been done on the Bernstein/Laplace side to deduce information on the coagulation-fragmentation side (see, for example [Ley03; MP04; DLP17; TV21]) but hardly the other way around. Our work is an example how one can "bootstrap" knowledge from both sides to gain more information.

The above conjecture, if true, would qualitatively confirm the infinite version of the prediction by Ben-Naim and Krapivsky in the weak sense without any rate. We conjecture that this is true in L^1 sense as well for any fixed time $t > T_{\text{gel}}$. We do not see evidence for what the rate of convergence should be.

II.3.1.1. Plan of the chapter. We outline our plan for the chapter. We start with some preliminaries (Section II.3.2) where we show Propositions II.3.1.2 and II.3.1.4. Then, we study the tail behavior of Flory solution and give the proof to Theorem II.3.1.5 in Section II.3.3. Lastly, we study the scaling limit of the tail of solution to (II.3.1.2) and give the some heuristics as well as conditional results towards confirming Conjecture II.3.1.6 in Section II.3.4.

II.3.2. Preliminaries

II.3.2.1. Equilibrium. In this subsection, we will give a proof of Proposition II.3.1.2.

Proof of Proposition II.3.1.2. We proceed by induction by noticing that this is a system of infinite algebraic equations and the equation for $\bar{\rho}_k$ depends on $\bar{\rho}_1, ..., \bar{\rho}_{k-1}$ as

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j(t)\rho_{k-j}(t) - km\rho_k(t) + \sum_{j=1}^{\infty}\varepsilon(k+j)\rho_{k+j}(t) - \frac{1}{2}\varepsilon k(k-1)\rho_k(t)\\
=&\frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j(t)\rho_{k-j}(t) - km\rho_k(t) + \varepsilon\Big(m - \sum_{j=1}^{k} j\rho_j(t)\Big) - \frac{1}{2}\varepsilon k(k-1)\rho_k(t)\,.
\end{aligned}$$

The base case $k = 1$ is verified by the following computation

$$0 = -m\bar{\rho}_1 + \varepsilon(m - \bar{\rho}_1)\,.$$

Therefore,

$$\bar{\rho}_1 = \frac{\varepsilon m}{m+\varepsilon}\,.$$

The rest of the proof is a standard induction argument. □

II.3.2.2. Singular limit. In this subsection, we will give a proof of Proposition II.3.1.4. Let us remind ourselves that for each $\varepsilon > 0$, ρ^ε solves equation (II.3.1.2), i.e. for each $k \in \mathbb{N}$,

$$\begin{aligned}
\partial_t\rho_k^\varepsilon(t) =&\frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j^\varepsilon(t)\rho_{k-j}^\varepsilon(t) - k\rho_k^\varepsilon(t)\sum_{j=1}^{\infty} j\,\rho_j^\varepsilon(t)\\
&+\sum_{j=1}^{\infty}\varepsilon(k+j)\rho_{k+j}^\varepsilon(t) - \frac{1}{2}\varepsilon k(k-1)\rho_k^\varepsilon(t)\,,
\end{aligned}$$

with initial value

$$\rho_k(0) = \rho_{0,k}\,.$$

By conservation of mass (see Theorem II.3.1.1),

$$\sum_{j=1}^{\infty} j\rho_j(t) \equiv m\,.$$

Therefore, we can re-write the above equation as

$$\begin{aligned}\partial_t \rho_k^\varepsilon(t) =& \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho_j^\varepsilon(t)\rho_{k-j}^\varepsilon(t) - mk\rho_k^\varepsilon(t) \\ &+ \varepsilon m - \varepsilon\sum_{j=1}^{k} j\rho_j^\varepsilon(t) - \frac{1}{2}\varepsilon k(k-1)\rho_k^\varepsilon(t)\,.\end{aligned} \tag{II.3.2.1}$$

Abstractly, for each $i \in \mathbb{N}$,

$$\partial_t \rho_k^\varepsilon(t) = F_i^\varepsilon(\rho_1^\varepsilon(t), \dots, \rho_k^\varepsilon(t))\,,$$

where for fixed $(x_1, \dots, x_{k-1})$, $F_k^\varepsilon(x_1, \dots, x_{k-1}, \cdot)$ is an affine function and, locally uniformly,

$$\lim_{\varepsilon \to 0} F_k^\varepsilon = F_k\,,$$

where

$$F_k(x_1, \dots, x_k) = \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,x_j x_{k-j} - mkx_k\,.$$

We use the following lemma, whose proof is a standard ODE exercise and is omitted.

Lemma II.3.2.1. *For each $\varepsilon \geqslant 0$, let $F^\varepsilon : \mathbb{R}^k \to \mathbb{R}^k$ be locally Lipschitz functions. Let y^ε be solutions to*

$$\frac{d}{dt} y^\varepsilon = F^\varepsilon(y^\varepsilon)\,,$$

with initial data $y^\varepsilon(0) = y_0^\varepsilon$. Suppose $F^\varepsilon \to F^0$ locally uniformly and $y_0^\varepsilon \to y_0^0$. Then, locally uniformly,

$$\lim_{\varepsilon \to 0} y^\varepsilon = y^0\,.$$

Proposition II.3.1.4 is an immediate consequence of the Lemma II.3.2.1. □

II.3.3. Tail behavior of Flory solution

For notational convenience, in this section, we always denote the Flory solution to the pure coagulation equation by $\rho = \rho^0$. We need a few preliminary results before proving Theorem II.3.1.5. To aid the reader, we cite the following classical lemma whose proof can be found in the classic book of Feller [Fel71].

Lemma II.3.3.1 ([Fel71, Lemma XV.1.4]). *Let μ be a probability measure on $\mathbb{N}$ and φ be the characteristic function of μ, i.e.,*

$$\varphi(\zeta) = \sum_{k \in \mathbb{N}} e^{i\zeta k}\mu_k\,.$$

There exist only three possibilities:

(1) *$|\varphi(\zeta)| < 1$ for all ζ.*

(2) $|\varphi(\zeta)| = 1$ *and* $|\varphi(\zeta)| < 1$ *for* $0 < \zeta < 2\pi/\ell$, *in which case* $\mu = \sum f_k \delta_k$, φ *is* $2\pi/\ell$*-periodic and* $|\{k \mid f_k > 0\}| \geqslant 2$. *Here,* ℓ *(called the span of* μ_k*) is the maximal number so that the initial clusters are concentrated on* $\ell\mathbb{N}$.

(3) $|\varphi(\zeta)| = 1$ *for all* ζ, *in which case* $\mu = \delta_k$ *for some* $k \in \mathbb{N}$.

Remark II.3.3.2. In this section, upon rescaling the lattice, without loss of generality, we assume that ρ_0 has span $\ell = 1$. Therefore, $\varphi(\rho_0)$ has period 2π. This will simplify our notations greatly.

Our strategy is to study the Bernstein-Fourier transform of the mass density. Consider for $z \in \overline{\mathbb{C}}_+$ where $\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$

$$\tilde{K}(z,t) = \sum_{k\in\mathbb{N}} (1 - e^{-kz}) k \rho_k(t)\,.$$

Denote by $g(t)$ the mass lost to infinity of the Flory solution. Thus, monotonically,

$$\begin{aligned} g(t) &= 0\,, \qquad t \leqslant T_{\text{gel}}\,,\\ g(t) &> 0\,, \qquad t > T_{\text{gel}}\,. \end{aligned}$$

By the weak formulation of the CF, we find $\tilde{K}$ solves the equation

$$\partial_t(\tilde{K} + g(t)) - (\tilde{K} + g(t))\partial_z \tilde{K} = 0\,.$$

Let $K(z,t) \stackrel{\text{def}}{=} \tilde{K}(z,t) + g(t)$. Then K solves the classical Burgers's equation in the right half of the complex plane

$$\partial_t K - K\partial_z K = 0\,, \tag{II.3.3.1}$$

with initial data

$$K(z,0) = K_0(z) = \sum_{k\in\mathbb{N}} (1 - e^{-kz}) k \rho_k(0)\,. \tag{II.3.3.2}$$

We want to study the characteristics of this equation. In particular, we will show that the inverse of the characteristic map at any time $t > 0$ is a diffeomorphism from $\mathbb{C}_+$ to its image. From there, we can study the Fourier transform of the Flory solution by a pullback along the characteristics and arrive at the rate of decay of the tail of the Flory solution.

Let $s(t; z, t_0) \in \bar{\mathbb{C}}_+$ to be the characteristic curve starting from z at time t_0. Along $s(t; z, t_0)$, K is constant, i.e., $K(s(t; z, t_0), t) = K(z, t_0)$. Thus, s is given by

$$s(t; z, t_0) = z - K(z, t_0)(t - t_0)\,. \tag{II.3.3.3}$$

When $t_0 = 0$, we write

$$s(t;z) \stackrel{\text{def}}{=} s(t; z, 0) \quad \text{and} \quad z(t) \stackrel{\text{def}}{=} s(t;z)\,.$$

Let $B_\varepsilon(z)$ be the ball of radius ε around the point z in the complex plane $\mathbb{C}$. Let $T_z \overset{\text{def}}{=} \inf\Big\{t \geqslant 0 \,\Big|\, s(t;z) \in \partial\,\mathbb{C}_+ \cup \{z = x + i0 \mid x \in \mathbb{R}\}\Big\}$. Thus, T_z is the first time the characteristic curve starting from z hits the imaginary axis or the real axis.

Lemma II.3.3.3. *Let $z \in \mathbb{C}_+ \backslash \{z = x + i0 \,|\, x \in \mathbb{R}\}$ and and $T_z^{\Im}$ be the first time the characteristic hits the imaginary axis. Then $T_z = T_z^{\Im}$. In other words, the characteristic must hit the imaginary axis first before hitting the real axis.*

Proof. Let $z = x + iy$ where $x \in (0, \infty)$ and $y \in \mathbb{R}\backslash\{0\}$. Furthermore, write $z(t) = x(t) + iy(t)$. Thus, $z(0) = z$. We rewrite the equation (II.3.3.3) for the characteristics starting from z at time 0.

$$\begin{aligned} s(t;z) &= s(t; z(t_0), t_0) = z(t_0) - K(z(t_0), t_0)(t - t_0) \\ &= z(t_0) - (t - t_0)\Big(\sum_{k\in\mathbb{N}} (1 - e^{-kz(t_0)})k\rho_k(t_0) + g(t_0)\Big) \\ &= x - (t - t_0)\Big(\sum_{k\in\mathbb{N}} (1 - e^{-kx(t_0)} \cos(ky(t_0)))k\rho_k(t_0) + g(t_0)\Big) \\ &\quad + i\Big(y - (t - t_0) \sum_{k\in\mathbb{N}} e^{-kx(t_0)} \sin(ky(t_0))k\rho_k(t_0)\Big)\,. \end{aligned} \tag{II.3.3.4}$$

As the characteristics are straight lines, we only need to confirm the following ratio

$$\frac{y - \operatorname{Im} s(t;z)}{x - \operatorname{Re} s(t;z)} < \frac{\operatorname{Im} z}{\operatorname{Re} z} = \frac{y}{x}\,, \tag{II.3.3.5}$$

as it will confirm that, in the complex plane, the characteristics is “flatter” than the line connecting 0 to z initially. Consider

$$y - ye^{-kx} \cos(ky) - xe^{-kx} \sin(ky) > y - (x + y)e^{-x+y} > 0\,.$$

This immediately implies inequality (II.3.3.5) when we rewrite it in terms of infinite sums. Our proof is finished. $\square$

Remark II.3.3.4. It is an informative exercise to prove the above fact using backward characteristics. It is, in fact, how we first made the observation.

Lemma II.3.3.5. *Let $\varepsilon > 0$. There exists M_ε such that for every $z \in \mathbb{C}_+ \setminus \bigcup_{k=0,1,\ldots} B_\varepsilon(i2k\pi)$ and for $0 \leqslant t \leqslant T_z$, we have*

$$\left|\frac{\partial s(t;z)}{\partial z}\right| \geqslant M_\varepsilon\,. \tag{II.3.3.6}$$

Proof. By 2π-periodicity, we only consider $z = x+iy$ for which $0 < y < 2\pi$. Recall that $s(t;z) = z - K_0(z)t$. Thus,

$$\begin{aligned}\left|\frac{\partial s(t;z)}{\partial z}\right| &= \left|1 - \partial_z K_0(z)t\right| = \left|1 - t\sum_{k\in\mathbb{N}} e^{-kz}k^2\rho_{0,k}\right| \\ &= \left|1 - t\sum_{k\in\mathbb{N}} e^{-kx}(\cos(ky) + i\sin(ky))k^2\rho_{0,k}\right|.\end{aligned} \tag{II.3.3.7}$$

For convenience, instead of looking at complex balls $B_\varepsilon(0) \cup B_\varepsilon(i2\pi)$, we look at complex squares $\Big\{[0,\varepsilon] + i\Big([0,\varepsilon] \cup [2\pi-\varepsilon, 2\pi]\Big)\Big\}$. The claim is true for the balls by the obvious equivalence. We break our analysis into two parts.

Case I: $x \geqslant \varepsilon$. By Lemma II.3.3.3 we know that $\operatorname{Re} s(T_z;z) = 0$. Therefore, by (II.3.3.4),

$$T_z = \frac{x}{\sum(1 - e^{-kx}\cos(ky))k\rho_{0,k}} \leqslant \frac{x}{\sum(1-e^{-kx})k\rho_{0,k}} = \frac{x}{K_0(x)}\,. \tag{II.3.3.8}$$

We then have that since K_0 is strictly concanve,

$$T_z \sum_{k\in\mathbb{N}} e^{-kx}k^2\rho_{0,k} \leqslant \frac{x\partial_x K_0(x)}{K_0(x)} < 1\,. \tag{II.3.3.9}$$

Substituting this into (II.3.3.7), we have that for $0 \leqslant t \leqslant T_z$,

$$\left|\frac{\partial s(t;z)}{\partial z}\right| \geqslant 1 - T_z\sum_{k\in\mathbb{N}} e^{-kx}k^2\rho_{0,k} > 0\,. \tag{II.3.3.10}$$

Furthermore, we note that

$$\lim_{x\to\infty}\frac{x\partial_x K_0(x)}{K_0(x)} = \lim_{x\to\infty}\frac{x\sum e^{-kx}k^2\rho_{0,k}}{\sum(1-e^{-kx})k\rho_{0,k}} = 0$$

and monotonically, as $x \searrow 0$,

$$\frac{x\partial_x K_0(x)}{K_0(x)} \nearrow 1\,. \tag{II.3.3.11}$$

Thus, for every $\varepsilon > 0$, there exists M_ε such that inequality (II.3.3.6) holds for $z \in \mathbb{C}_+ \cap \{x \geqslant \varepsilon\}$.

Case II: $0 < x \leqslant \varepsilon \leqslant y \leqslant 2\pi - \varepsilon$. We re-consider

$$\begin{aligned}T_z\sum_{k\in\mathbb{N}} e^{-kx}k^2\rho_{0,k} &= \frac{x\partial_x K_0(x)}{\sum(1-e^{-kx}\cos(ky))k\rho_{0,k}} \\ &= \frac{x\partial_x K_0(x)}{K_0(x) + \sum e^{-kx}(1-\cos(ky))k\rho_{0,k}}.\end{aligned}$$

By Lemma II.3.3.1, there exists a constant $f_\varepsilon > 0$ such that

$$\sum_{k\in\mathbb{N}} e^{-kx}(1-\cos(ky))k\rho_{0,k} \geqslant f_\varepsilon\,.$$

Therefore,

$$T_z \sum_{k\in\mathbb{N}} e^{-kx} k^2 \rho_{0,k} \leqslant \frac{x\partial_x K_0(x)}{K_0(x)+f_\varepsilon} \leqslant 1 - M_\varepsilon\,,$$

for some $M_\varepsilon > 0$. Substituting this into (II.3.3.10), we arrive at our claim. ☐

A closer look at the above lemma reveals that one can get a quantiative lower bound for $|\partial s(t;z)/\partial z|$ when $z \to k2\pi i$, $k = 0, 1, \dots$.

Lemma II.3.3.6. *There exists $\delta > 0$ and $C > 0$ such that for $z \in B_\delta(k2\pi i)$, $k = 0, 1, \dots,$ we have*

$$\left|\frac{\partial s(t;z)}{\partial z}\right| \geqslant C|z|\,. \tag{II.3.3.12}$$

Furthermore, δ and C depend on the initial data ρ_0.

Proof. Without loss of generality, we only focus on the situation $z \to 0$. As done previously, instead of showing for the typical Euclidean norm, we provide the proof using the max norm $|z|_\infty = \max(|x|, |y|)$ on the right hand side (and leave the Euclidean norm on the left hand side) as these norms are equivalent. We divide the proof into two cases.

Case I: $0 \leqslant y < x$. We analyze (II.3.3.11) more carefully. By Taylor's theorem, there exist $h_1, h_2 \in (0,1)$ such that

$$\frac{K_0(x) - x\partial_x K_0(x)}{K_0(x)} = -\frac{1}{2}\frac{x^2\partial_x^2 K_0(h_1 x)}{x\partial_x K_0(h_2 x)} \geqslant Cx\,,$$

where $C > 0$ only depends on ρ_0. Re-visiting (II.3.3.10), there exists δ such that for all $x \in (0,\delta)$, we have

$$\left|\frac{\partial s(t;z)}{\partial z}\right| \geqslant 1 - \frac{x\partial_x K_0(x)}{K_0(x)} \geqslant Cx = C|z|_\infty\,.$$

Case II: $0 \leqslant x < y$.

Step 1: The difficulty in this case that x could be 0 and hence T_z would be arbitrarily small. Let $\overline{T}_x = T_z$ where $z = x + 0i$. Thus, from (II.3.3.8), for $z = x + iy$,

$$T_z \leqslant \overline{T}_x\,.$$

From (II.3.3.7), we have that

$$\begin{aligned}\left|\frac{\partial s(t;z)}{\partial z}\right| &= \left|1 - t\sum_{k\in\mathbb{N}} e^{-kx}(\cos(ky) + i\sin(ky))k^2\rho_{0,k}\right| \\ &\geqslant \frac{1}{2}\left(\left|1 - t\sum_{k\in\mathbb{N}} e^{-kx}\cos(ky)k^2\rho_{0,k}\right| + t\left|\sum_{k\in\mathbb{N}} e^{-kx}\sin(ky)k^2\rho_{0,k}\right|\right).\end{aligned}$$

If $0 \leqslant t \leqslant \overline{T}_x/2$, by (II.3.3.9) and (II.3.3.11), we have

$$1 - t\sum_{k\in\mathbb{N}} e^{-kx}\cos(ky)k^2\rho_{0,k} \geqslant 1 - \frac{\overline{T}_x}{2}\sum_{k\in\mathbb{N}} e^{-kx}k^2\rho_{0,k} \geqslant \frac{1}{2} \geqslant |z|_\infty\,. \tag{II.3.3.13}$$

Step 2: Furthermore, we also observe that as z approaches the imaginary axis, $\overline{T}_x \to 0$ as well. Therefore, there exists a constant r_0 such that for $0 \leqslant x \leqslant r_0$,

$$\overline{T}_x \leqslant \frac{m_2(0)}{100}\,.$$

Using (II.3.3.13) once more, we can then assume that $y > x > r_0$ for the rest of this proof. Therefore,

$$\overline{T}_x \geqslant \frac{m_2(0)}{100}\,.$$

Step 3: When $\overline{T}_x/2 \leqslant t \leqslant \overline{T}_x$, let L be such that

$$\sum_{k=L}^{\infty} k^2\rho_{0,k} \leqslant \frac{r_0}{100}\sum_{k=1}^{\infty} k^3\rho_{0,k}\,.$$

Choose δ such that $\delta L < \pi/2$. If $\delta \leqslant r_0$, then we go back to Step 2. Therefore, we can assume that $\delta > y > x > r_0$. By our choice of δ, we ensure that for $k = 1, ..., K$, it is true that $\sin(ky) \geqslant ky/2$. Therefore,

$$\begin{aligned}
&\frac{\overline{T}_x}{2}\Big|\sum_{k\in\mathbb{N}} e^{-kx}\sin(ky)k^2\rho_{0,k}\Big| \\
&= \frac{\overline{T}_x}{2}\Big|\sum_{k=1}^{L} e^{-kx}\sin(ky)k^2\rho_{0,k} + \sum_{k>L} e^{-kx}\sin(ky)k^2\rho_{0,k}\Big| \\
&\geqslant \frac{\overline{T}_x}{2}\Big(\sum_{k=1}^{L} e^{-kx}\sin(ky)k^2\rho_{0,k} - \sum_{k>L} e^{-kx}k^2\rho_{0,k}\Big) \\
&\geqslant \frac{\overline{T}_x}{2}\Big(\frac{y}{2}\sum_{k=1}^{L} e^{-kx}k^3\rho_{0,k} - \frac{r_0}{100}\sum_{k=1}^{L} k^3\rho_{0,k}\Big) \\
&\geqslant \frac{\overline{T}_x}{2}\Big(\frac{y}{2}\sum_{k=1}^{L} e^{-\pi/2}k^3\rho_{0,k} - \frac{y}{100}\sum_{k=1}^{L} k^3\rho_{0,k}\Big) \\
&\geqslant Cy = C|z|_\infty\,,
\end{aligned} \tag{II.3.3.14}$$

where C is independent of ε. From (II.3.3.13) and (II.3.3.14), we deduce that

$$\Big|\frac{\partial s(t;z)}{\partial z}\Big| \geqslant 1 - \frac{x\partial_x K_0(x)}{K_0(x)} \geqslant Cx = C|z|_\infty\,,$$

finishing the proof. □

Remark II.3.3.7. It is true that $K \in C((0,\infty); C^\infty(\mathbb{C}_+))$ as it is the Bernstein transform of $\rho^0(t)$.

LEMMA II.3.3.8. *Let $t > 0$. Then the characteristic map $s(t;\cdot) : s^{-1}(t;\mathbb{C}_+) \to \mathbb{C}_+$ is a diffeomorphism.*

PROOF. By definition, $s(t;\cdot)$ is onto. By the formula (II.3.3.3), s is continuous. We then want to show one-to-one property. Furthermore, the characteristics $s(t;z)$ satisfies the equation

$$\frac{\partial s(t;z)}{\partial t} = -K(s(t;z),t)\,.$$

Therefore, by Remark II.3.3.7 uniqueness of ODE, $s(t;\cdot)$ is one-to-one. Thus, for every $\tilde{z} \in \mathbb{C}_+$, we can find a unique z such that

$$\tilde{z} = s(t;z)\,.$$

We can then define the inverse function $s^{-1}(t;\cdot) : \mathbb{C}_+ \to s^{-1}(t;\mathbb{C}_+)$ as

$$s^{-1}(t;\tilde{z}) = z\,. \tag{II.3.3.15}$$

Using (II.3.3.3), we can deduce that for $s^{-1}(t;\cdot)$ is continuous. Hence, $s(t;\cdot)$ is a homeomorphism. As s and s^{-1} are both differentiable, the proof is finished. $\square$

REMARK II.3.3.9. A consequence of Lemmas II.3.3.5 and II.3.3.8 is that for every point $\tilde{z} \in \mathbb{C}_+ \backslash \bigcup_{k=0,1,\dots} B_\varepsilon(i2k\pi)$, there exists a $\delta = \delta_\varepsilon$ such that

$$s^{-1}(t;\tilde{z}) \in \mathbb{C}_+ \backslash \bigcup_{k=0,1,\dots} B_\delta(i2k\pi)\,,$$

and

$$\left|\frac{\partial s^{-1}(t;\tilde{z})}{\partial \tilde{z}}\right| \leqslant \frac{1}{M_\delta}\,,$$

for some $M_\delta > 0$.

Next, we would like to study the regularity of K along the imaginary axis. To this end, we need to understand the interplay between the relationship between the domain and image of the characteristics $s(t;z)$ at a given time t.

PROPOSITION II.3.3.10. *Let $t > 0$, then $K(\cdot,t) \in C^{0,1/2}(\overline{\mathbb{C}}_+)$ and that there exists a constant C independent of t and ε such that*

$$\sup_{z_1,z_2\in\overline{\mathbb{C}}_+} \frac{|K(\tilde{z}_1,t) - K(\tilde{z}_2,t)|}{|\tilde{z}_1 - \tilde{z}_2|^{1/2}} \leqslant C\,.$$

PROOF. Let $\varepsilon > 0$ be a small constant to be chosen later and $\tilde{z} \in S_\varepsilon \stackrel{\text{def}}{=} \{x + iy \mid x \geqslant \varepsilon\}$. We could then find $\delta = \delta_\varepsilon > 0$ such that $s^{-1}(S_\varepsilon) \subseteq S_\delta$.

Furthermore, by the Remark II.3.3.7, singularities (if they exist) can only occur on the imaginary axis $\{z = iy\}$. By 2π periodicity, we only consider the case near the imaginary axes. Letting $\tilde{z} = s(t;z)$ and rewriting (II.3.3.3), we get

$$\tilde{z} = z - t\sum_{k\in\mathbb{N}}(1 - e^{-kz})k\rho_{0,k}\,.$$

Rearranging terms, it follows that

$$\begin{aligned}\tilde{z} + mt &= z + \sum_{k\in\mathbb{N}} e^{-kz} k\rho_{0,k} t \\ &= z + \sum_{k\in\mathbb{N}} \Big(1 - kz + \frac{(kz)^2}{2}\Big) k\rho_{0,k} t + \sum_{k\in\mathbb{N}} \sum_{n=3}^{\infty} \frac{(-kz)^n}{n!} k\rho_{0,k} t\,.\end{aligned}$$

Thus,

$$\tilde{z} = z - zm_2(0)t + \frac{m_3(0)}{2} z^2 t + R(z)t\,, \tag{II.3.3.16}$$

where

$$R(z) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{N}} \sum_{n=3}^{\infty} \frac{(-kz)^n}{n!} k\rho_{0,k}. \tag{II.3.3.17}$$

We divide our analysis into two different regions.

Step 4: First, fix $y_0 \in \bigcup_{k=0,1,\ldots} (i2k\pi + \varepsilon, i2(k+1)\pi - \varepsilon)$ and consider the sequences $\{\tilde{z}_j^1, \tilde{z}_j^2\} \in \mathbb{C}_+$ such that

$$\lim_{j\to\infty} z_j^1 = \lim_{j\to\infty} z_j^2 = iy_0\,.$$

For $l = 1, 2$, write

$$z_j^l = s^{-1}(t; \tilde{z}_j^l)\,.$$

By Remark II.3.3.9, there exists $M_\varepsilon \to 0$ as $\varepsilon \to 0$ such that for large enough j, k,

$$\begin{aligned}\frac{|K(\tilde{z}_j^1, t) - K(\tilde{z}_k^2, t)|}{|\tilde{z}_j^1 - \tilde{z}_k^2|} &\leqslant \frac{|K_0(z_j^1) - K_0(z_k^2)|}{M_\varepsilon |z_j^1 - z_k^2|} \\ &\leqslant C_\varepsilon |\partial_z K_0(z_j^1)| \leqslant \tilde{C}_\varepsilon\,,\end{aligned}$$

where $\tilde{C}_\varepsilon \to \infty$ as $\varepsilon \to 0$. Therefore,

$$\frac{|K(iy_0, t) - K(\tilde{z}_i^2, t)|}{|iy_0 - \tilde{z}_k^2|} = \lim_{j\to\infty} \frac{|K(\tilde{z}_j^1, t) - K(\tilde{z}_k^2, t)|}{|\tilde{z}_j^1 - \tilde{z}_k^2|} \leqslant \tilde{C}_\varepsilon\,,$$

showing that $K \in \mathrm{Lip}\Big(\overline{\mathbb{C}}_+ \setminus \bigcup_{k=0,1,\ldots} B_\varepsilon(i2k\pi)\Big)$.

Step 5: By the previous step, we see that the singularity points are $\tilde{z} = k2\pi$ where $k = 0, 1, \ldots$. By 2π periodicity, we only need to analyze when $\tilde{z} = 0$. Consider the sequences $\{\tilde{z}_j^1, \tilde{z}_j^2\} \in \mathbb{C}_+$ such that

$$\lim_{j\to\infty} z_j^1 = \lim_{j\to\infty} z_j^2 = 0\,.$$

Again, for $l = 1, 2$, write

$$z_j^l = s^{-1}(t; \tilde{z}_j^l)\,.$$

By Lemma II.3.3.6, we have

$$|\tilde{z}_j^1 - \tilde{z}_k^2| \geqslant C \max(|z_j^1|, |z_k^2|)|z_j^1 - z_k^2| \geqslant C|z_j^1 - z_k^2|^2\,.$$

$$
\begin{aligned}
\frac{|K(\tilde{z}_j^1,t)-K(\tilde{z}_k^2,t)|}{|\tilde{z}_j^1-\tilde{z}_k^2|^{1/2}} &\leqslant \frac{|K_0(z_j^1)-K_0(z_k^2)|}{C|z_j^1-z_k^2|} \\
&\leqslant \tilde{C}\,,
\end{aligned}
$$

where $\tilde{C}$ is independent of ε. The proof is complete. □

REMARK II.3.3.11. In the proof above, we actually see that K is Lipschitz away from the points $\tilde{z} = i2k\pi$, where $k = 0, 1, \ldots$.

PROOF OF THEOREM II.3.1.5. In the case of monomers, $\mu = \delta_1$, Theorem II.3.1.5 is true because of the explicit formula for the solution (II.3.1.6). Thus, we assume that $\mu = \sum f_k \delta_k$ where f_k is non-zero for at least two indices. Recall Remark II.3.3.2 that we assume the span ℓ of ρ_0 to be 1.

Step 1: Recall that for $y > 0$, there exists a unique $z \in \mathbb{C}_+$ such that

(II.3.3.18) $$iy = s(t;z) = z - tK_0(z) \quad \text{and} \quad K(iy,t) = K_0(z)\,.$$

Therefore, there exists a curve $\Gamma \in \mathbb{C}_+$ such that $s(t;\Gamma) = \{z = iy \,|\, y \in [0, 2\pi]\}$. We have that

$$K(iy,t) = \sum_{k\in\mathbb{N}} (1 - e^{-iky}) k\rho_k(t)\,.$$

By Fourier-Laplace inversion formula and a change of variable using the relations (II.3.3.18), we have

(II.3.3.19)
$$
\begin{aligned}
tk\rho_k(t) &= \frac{t}{2\pi}\int_0^{2\pi} \sum_{j\in\mathbb{N}} e^{i(k-j)y} j\rho_j(t)\,dy = -\frac{t}{2\pi}\int_0^{2\pi} e^{iky} K(iy,t)\,dy \\
&= \frac{t}{ik2\pi}\int_0^{2\pi} e^{iky}\partial_y K(iy,t)\,dy = \frac{t}{ik2\pi}\int_\Gamma e^{k(z-tK_0(z))}\partial_z K_0(z)\,dz \\
&= -\frac{1}{ik2\pi}\int_\Gamma e^{k(z-tK_0(z))}\Big((1 - t\partial_z K_0(z)) - 1\Big)\,dz \\
&= -\frac{1}{k2\pi}\int_0^{2\pi} e^{iky}\,dy + \frac{1}{ik2\pi}\int_\Gamma e^{k(z-tK_0(z))}\,dz\,.
\end{aligned}
$$

The first integral vanishes as the integrand is 2π-periodic. We want to analyze the second integral. Let $\tilde{\Gamma}$ be the directed straight line that connects $i2\pi$ to 0. Note that in the y-direction, the characteristics obeys the following periodicity property

$$s(t; iy + x) = i(2\pi + y) + s(t; i(2\pi + y) + x)\,.$$

Let β the closed path that contains Γ, $\tilde{\Gamma}$, and two straight lines that connect Γ to $\tilde{\Gamma}$. By the periodicity property of the characteristics, the integrals of $\exp(k(z - tK_0(z)))$ over the two straight horizontal lines cancel each other (see Figure II.3.3.1). By Cauchy's integral formula, we have

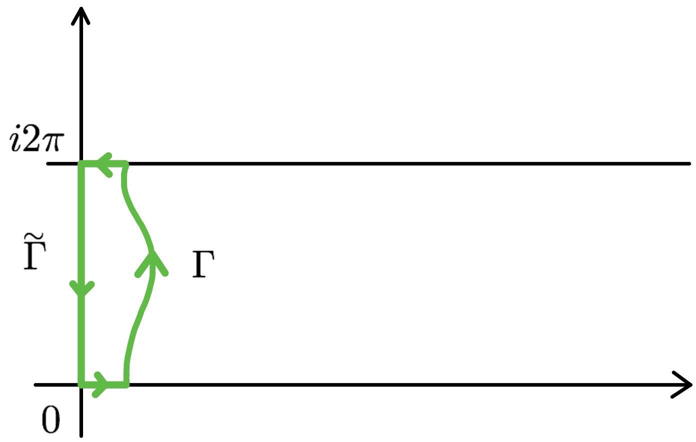

FIGURE II.3.3.1. Contour integral

$$\int_\Gamma \exp(k(z - tK_0(z)))\, dz = i\int_0^{2\pi} \exp(k(i\lambda - tK_0(i\lambda)))\, d\lambda\,.$$

Here, we parametrize $\tilde{\Gamma}$ by $i\lambda$ where $\lambda \in [0, 2\pi]$. Using the formula (II.3.3.2) for K_0, we have

$$\left|\int_\Gamma e^{kz}\, dz\right| \leqslant \int_0^{2\pi} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda)k\rho_{0,k}\Big)\Big)\, d\lambda\,.$$

Step 2: Let L, δ be constants (depending on ρ_0) such that

$$\sum_{k=1}^{L} k\rho_{0,k} \geqslant 10\sum_{k=L}^{\infty} k\rho_{0,k} \quad \text{and} \quad \delta L \leqslant \frac{\pi}{4}\,.$$

Note that with the choices of L and δ, we have $\cos(k\lambda) \geqslant 1/2$ for $k \leqslant L$ and $0 \leqslant \lambda \leqslant \delta$. Thus,

$$\begin{aligned}
\left|\int_\Gamma e^{-kz}\, dz\right| &\leqslant \int_0^{2\pi} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda)k\rho_{0,k}\Big)\Big)\, d\lambda \\
&\leqslant \int_{[0,\delta]\cup[2\pi-\delta,2\pi]} \exp\Big(-tkm + \frac{tk}{2}\sum_{k=1}^{L} \cos(k\lambda)k\rho_{0,k}\Big)\, d\lambda \\
&\quad + \int_\delta^{2\pi-\delta} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda)k\rho_{0,k}\Big)\Big)\, d\lambda \\
&\leqslant \int_{[0,\delta]\cup[2\pi-\delta,2\pi]} \exp\Big(-tkm + \frac{tk}{2}\sum_{k=1}^{L} 2(1-(k\lambda)^2)k\rho_{0,k}\Big)\, d\lambda \\
&\quad + \int_\delta^{2\pi-\delta} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda)k\rho_{0,k}\Big)\Big)\, d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{[0,\delta]\cup[2\pi-\delta,2\pi]} \exp\Big(-tkm + tk\sum_{k=1}^{\infty} k\rho_{0,k} - tk\sum_{k=1}^{L}(k\lambda)^2 k\rho_{0,k}\Big)\, d\lambda \\
&\quad + \int_{\delta}^{2\pi-\delta} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda) k\rho_{0,k}\Big)\Big)\, d\lambda \\
&\leqslant \frac{C}{\sqrt{kt}} + \int_{\delta}^{2\pi-\delta} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda) k\rho_{0,k}\Big)\Big)\, d\lambda\,.
\end{aligned}
$$

By Lemma II.3.3.1 we know that there exists an $\eta = \eta_\delta > 0$ such that for $\lambda \in [\delta, 2\pi - \delta]$, it is true that

$$
m - \sum_{k\in\mathbb{N}} \cos(k\lambda) k\rho_{0,k} \geqslant \eta\,.
$$

Therefore, by a change of variable $\tilde{\lambda} = k\lambda$ and integrating in terms of $\tilde{\lambda}$, we have that

$$
\int_{\delta}^{2\pi-\delta} \exp\Big(tk\Big(-m + \sum_{k\in\mathbb{N}} \cos(k\lambda) k\rho_{0,k}\Big)\Big)\, d\lambda \leqslant \frac{Ce^{-tk\eta}}{k}\,.
$$

Combining this with equation (II.3.3.19), we get

$$
\rho_k(t) \leqslant \frac{C}{(tk)^{5/2}}\,. \tag{II.3.3.20}
$$

Step 3: Lastly, observe that for $0 \leqslant t \leqslant T_{\text{gel}}/2 = 1/(2m_2(0))$,

$$
|\partial_z s(t;z)| = |1 - t\partial_z K_0(z)| = \Big|1 + t\sum_{k\in\mathbb{N}} e^{kz} k^2 \rho_{0,k}\Big| \geqslant \frac{T_{\text{gel}}}{2}\,.
$$

Therefore, $s(t;z)$ is a diffeomophism and $K(z,t)$ is smooth. By Fourier-Laplace inversion formula and integration by parts,

$$
k\rho_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{iky} K(iy,t)\, dy = -\frac{1}{4\pi^2 k^2}\int_0^{2\pi} e^{iky}\partial_y^2 K(iy,t)\, dy\,.
$$

Since the integrand in the last integral is bounded, there exists a constant $C > 0$, independent of t, such that

$$
\rho_k(t) \leqslant \frac{C}{k^3}\,. \tag{II.3.3.21}
$$

Combining (II.3.3.20) and (II.3.3.21), estimate (II.3.1.7) follows immediately.
$\square$

II.3.4. Scaling limit of tails

In this section, we will discuss Conjecture II.3.1.6. We outline a very promising way to prove this conjecture and discuss where the bottleneck is (see Remark II.3.4.5).

Recall that ρ^0 is the Flory solution to the pure coagulation equation. We then define $\bar{c}^\varepsilon(s,t) : [0,\infty)^2 \to \mathbb{R}$,

$$\bar{c}^\varepsilon(s,t) \stackrel{\text{def}}{=} \varepsilon^{-2}\rho^0_{\lfloor s/\varepsilon \rfloor}(t)\,.$$

Observe that $s\bar{c}^\varepsilon(s,t)\,ds \xrightharpoonup{*} (m-g(t))\delta_0$ as ε tends to 0. Hence, to prove Conjecture II.3.1.6, it is enough to show

$$s(c^\varepsilon(s,t)-\bar{c}^\varepsilon(s,t))\,ds \xrightharpoonup{*} e^{-s/g(t)}\,ds\,. \tag{II.3.4.1}$$

First, we define, for each $t \in (0,\infty)$, functions $\tilde{c}^\varepsilon(t), \hat{c}^\varepsilon(t) : \varepsilon\mathbb{N} \to \mathbb{R}$,

$$\tilde{c}^\varepsilon_{\varepsilon k}(t) \stackrel{\text{def}}{=} \frac{\rho^\varepsilon_k(t)}{\varepsilon} \quad \text{and} \quad \hat{c}^\varepsilon_{\varepsilon k}(t) \stackrel{\text{def}}{=} \frac{\rho^0_k(t)}{\varepsilon}\,.$$

We note that

$$\tilde{c}^\varepsilon_{\varepsilon k}(t) = \int_{\varepsilon k}^{\varepsilon k+\varepsilon} c^\varepsilon(s,t)\,ds = \varepsilon c^\varepsilon(\varepsilon k,t)\,, \tag{II.3.4.2a}$$

and

$$\hat{c}^\varepsilon_{\varepsilon k}(t) = \int_{\varepsilon k}^{\varepsilon k+\varepsilon} \bar{c}^\varepsilon(s,t)\,ds = \varepsilon \bar{c}^\varepsilon(\varepsilon k,t)\,. \tag{II.3.4.2b}$$

Then, we have

$$\begin{aligned}
\partial_t \tilde{c}^\varepsilon_{\varepsilon k}(t) =& \frac{1}{\varepsilon}\Big(\frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\,\rho^\varepsilon_j(t)\rho^\varepsilon_{k-j}(t) - k\rho^\varepsilon_k(t)\sum_{j=1}^{\infty} j\,\rho^\varepsilon_j(t)\Big)\\
&+\frac{1}{\varepsilon}\Big(\varepsilon\sum_{j=1}^{\infty}(k+j)\rho^\varepsilon_{k+j}(t) - \frac{\varepsilon}{2}k(k-1)\rho^\varepsilon_k(t)\Big)\\
=&\frac{1}{2\varepsilon}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)\tilde{c}^\varepsilon_{\varepsilon j}(t)\tilde{c}^\varepsilon_{\varepsilon k-\varepsilon j}(t) - k\tilde{c}^\varepsilon_{\varepsilon k}(t)\sum_{j=1}^{\infty}\varepsilon j\,\tilde{c}^\varepsilon_{\varepsilon j}(t)\\
&+\sum_{j=1}^{\infty}(\varepsilon k+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon k+\varepsilon j}(t) - \frac{1}{2\varepsilon}\varepsilon k(\varepsilon k-\varepsilon)\tilde{c}^\varepsilon_{\varepsilon k}(t)\,.
\end{aligned} \tag{II.3.4.3}$$

Substitute (II.3.4.2) into (II.3.4.3), we get

(II.3.4.4)
$$
\begin{aligned}
&\varepsilon\partial_t c^\varepsilon(\varepsilon k,t) = \\
&\quad = \frac{\varepsilon}{2}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)c^\varepsilon(\varepsilon j,t)c^\varepsilon(\varepsilon k-\varepsilon j,t) - \varepsilon k c^\varepsilon(\varepsilon k,t)\sum_{j=1}^{\infty}\varepsilon^2 j\, c^\varepsilon(\varepsilon j,t) \\
&\qquad + \varepsilon\sum_{j=1}^{\infty}(\varepsilon k+\varepsilon j)c^\varepsilon(\varepsilon k+\varepsilon j,t) - \frac{1}{2}\varepsilon k(\varepsilon k-\varepsilon)c^\varepsilon(\varepsilon k,t)\,.
\end{aligned}
$$

Of course, for every $\varepsilon > 0$, $m = \sum_{j=1}^{\infty}\varepsilon^2 j\, c^\varepsilon_{\varepsilon j}(t)$. However, as the mass splits when $\varepsilon \to 0$, we expect that there exists a function $\bar{c} : [0,\infty) \to \mathbb{R}$ so that $sc^\varepsilon(s,t)ds \overset{*}{\rightharpoonup} (m-g(t))\delta_0 + \bar{c}(s,t)\,ds$, where $g(t) = \int s\bar{c}(s,t)\,ds$ is the lost mass of the Flory solution. Heuristically but incorrectly ignoring $(m-g(t))\delta_0$, we could blindly intepret $\sum_{j=1}^{\infty}\varepsilon^2 j\, c^\varepsilon_{\varepsilon j}(t)$ as the total mass of giant particles, which is approximated by $g(t)$. Therefore, neglecting the interaction between giant particles and small particles, if such a function $\bar{c}$ exists, then by sending $\varepsilon \to 0$ in equation (II.3.4.4), it must satisfy the following equation

$$
\begin{aligned}
0 =& \frac{1}{2}\int_0^s \hat{s}(s-\hat{s})\bar{c}(\hat{s},t)\bar{c}(s-\hat{s},t)\,d\hat{s} - sg(t)\bar{c}(s,t) \\
&+ \int_s^\infty \hat{s}\bar{c}(\hat{s},t)\,d\hat{s} - \frac{s^2}{2}\bar{c}(s,t)
\end{aligned}
$$

whose solution is

$$
\bar{c}(s,t) = \frac{e^{-s/g(t)}}{s}\,.
$$

Our goal is to make these heuristics rigorous.

Lemma II.3.4.1. *For every $\varepsilon < \varepsilon_0$, let $h^\varepsilon = \tilde{c}^\varepsilon - \hat{c}^\varepsilon$. Then,*

$$
|h^\varepsilon_{\varepsilon k}| \leqslant \frac{1 + Ck^{-1/2}}{k + \frac{\varepsilon k(k-1)}{2m}}\,, \tag{II.3.4.5}
$$

and

$$
|\partial_t h^\varepsilon_{\varepsilon k}| \leqslant M\,. \tag{II.3.4.6}
$$

for some constants $C, M > 0$, independent of ε.

Proof. We divide the proof of the theorem into a few steps.

Step 1: We first show the upper bound. Note that $h^\varepsilon(0) = 0$. We proceed by induction. First, for $k = 1$, we have that

$$
\partial_t h^\varepsilon_\varepsilon = -mh^\varepsilon_\varepsilon(t) + \sum_{j=1}^{\infty}(\varepsilon + \varepsilon j)\tilde{c}^\varepsilon_{\varepsilon+\varepsilon j}\,.
$$

Thus,

$$0 \leqslant h^\varepsilon_\varepsilon(t) = e^{-mt}\int_0^t e^{ms}\sum_{j=1}^\infty(\varepsilon+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon+\varepsilon j}(s)\,ds \leqslant 1-e^{-mt}\,.$$

Therefore,

$$h^\varepsilon_\varepsilon \leqslant 1 = \frac{1}{\varepsilon 1/\varepsilon}\,.$$

Now, suppose it is true for $j = 1,\dots,k-1$ that

$$h^\varepsilon_{\varepsilon j} \leqslant \frac{1}{j}\,. \tag{II.3.4.7}$$

We want to show that this inequality is true for $j = k$. Consider

$$\begin{aligned}
\partial_t h^\varepsilon_{\varepsilon k} &= \frac{1}{2\varepsilon}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)(\tilde{c}^\varepsilon_{\varepsilon j}\tilde{c}^\varepsilon_{\varepsilon k-\varepsilon j} - \hat{c}^\varepsilon_{\varepsilon j}\hat{c}^\varepsilon_{\varepsilon k-\varepsilon j}) - kmh^\varepsilon_{\varepsilon k}\\
&\quad + \sum_{j=1}^\infty(\varepsilon k+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon k+\varepsilon j} - \frac{1}{2\varepsilon}\varepsilon k(\varepsilon k-\varepsilon)\tilde{c}^\varepsilon_{\varepsilon k}\\
&= \frac{1}{2\varepsilon}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)(\tilde{c}^\varepsilon_{\varepsilon j}h^\varepsilon_{\varepsilon k-\varepsilon j} + \hat{c}^\varepsilon_{\varepsilon k-\varepsilon j}h^\varepsilon_{\varepsilon j}) - kmh^\varepsilon_{\varepsilon k}\\
&\quad + \sum_{j=1}^\infty(\varepsilon k+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon k+\varepsilon j} - \frac{1}{2\varepsilon}\varepsilon k(\varepsilon k-\varepsilon)\tilde{c}^\varepsilon_{\varepsilon k}\\
&\leqslant \frac{1}{2\varepsilon}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)\Big(\frac{\tilde{c}^\varepsilon_{\varepsilon j}}{k-j} + \frac{\hat{c}^\varepsilon_{\varepsilon k-\varepsilon j}}{j}\Big) - \frac{1}{2}\varepsilon k(k-1)h^\varepsilon_{\varepsilon k}\\
&\quad - kmh^\varepsilon_{\varepsilon k} + \sum_{j=1}^\infty(\varepsilon k+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon k+\varepsilon j} - \frac{1}{2}\varepsilon k(k-1)\hat{c}^\varepsilon_{\varepsilon k}\\
&\leqslant \frac{1}{2}\sum_{j=1}^\infty \varepsilon(j\tilde{c}^\varepsilon_{\varepsilon j} + j\hat{c}^\varepsilon_{\varepsilon j}) - kmh^\varepsilon_k\\
&= m - \Big(km + \frac{1}{2}\varepsilon k(k-1)\Big)h^\varepsilon_{\varepsilon k}\,.
\end{aligned} \tag{II.3.4.8}$$

Thus, by Gronwall's inequality,

$$h^\varepsilon_{\varepsilon k}(t) \leqslant \frac{1}{k+\frac{\varepsilon k(k-1)}{2m}}\,, \tag{II.3.4.9}$$

as desired.

Step 2: Let $p = 1/2$. We now show the lower bound. It is true for the base case $k = 1$ as $h^\varepsilon_\varepsilon(t) \geqslant 0$. On the one hand, suppose that

$$h^\varepsilon_{\varepsilon j} \geqslant \frac{-1-Cj^{-1/2}}{j+\frac{\varepsilon j(j-1)}{2m}}$$

is true for $j \leqslant k-1$. Again, from a similar computation as above, we have

$$\begin{aligned}\partial_t h^\varepsilon_{\varepsilon k} &= \frac{1}{2\varepsilon}\sum_{j=1}^{k-1}\varepsilon^2 j(k-j)(\tilde{c}^\varepsilon_{\varepsilon j}h^\varepsilon_{\varepsilon k-\varepsilon j}+\hat{c}^\varepsilon_{\varepsilon k-\varepsilon j}h^\varepsilon_{\varepsilon j}) - kmh^\varepsilon_{\varepsilon k}\\ &\quad+\sum_{j=1}^{\infty}(\varepsilon k+\varepsilon j)\tilde{c}^\varepsilon_{\varepsilon k+\varepsilon j} - \frac{1}{2\varepsilon}\varepsilon k(\varepsilon k-\varepsilon)\tilde{c}^\varepsilon_{\varepsilon k}\\ &\geq -m - kmh^\varepsilon_{\varepsilon k} - \frac{1}{2}\varepsilon k(k-1)h^\varepsilon_{\varepsilon k} - \frac{C}{2k^{1/2}}\,.\end{aligned}\tag{II.3.4.10}$$

The last inequality follows from the tail bound of Flory solution, i.e., Theorem II.3.1.5. By Gronwall's inequality once again, we have

$$h^\varepsilon_{\varepsilon k} \geqslant \frac{-1-Ck^{-1/2}}{k+\frac{\varepsilon k(k-1)}{2m}}\,,$$

as desired. Estimate (II.3.4.6) also follows immediately after using the bound for $|h^\varepsilon_k|$ in (II.3.4.8) and (II.3.4.10). □

We next perform the discrete Laplace transform for $\tilde{c}^\varepsilon$ and $\hat{c}^\varepsilon$. Define

$$F^\varepsilon(x,t)=\sum_{k\in\mathbb{N}}e^{-x\varepsilon k}\tilde{c}^\varepsilon_{\varepsilon k}(t)\,,\quad \hat{F}^\varepsilon(x,t)=\sum_{k\in\mathbb{N}}e^{-x\varepsilon k}\hat{c}^\varepsilon_{\varepsilon k}(t)\,.$$

The resulting equations for F^ε and $\hat{F}^\varepsilon$ are

$$\begin{aligned}\partial_t F^\varepsilon(x,t) &= \frac{1}{2\varepsilon}(\partial_x F^\varepsilon)^2+\frac{m}{\varepsilon}\partial_x F^\varepsilon\\ &\qquad -\frac{1}{2}\sum_{k=2}^{\infty}e^{-x\varepsilon k}k(k-1)\rho^\varepsilon_k+\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}e^{-x\varepsilon j}k\rho^\varepsilon_k\\ &= \frac{1}{2\varepsilon}(\partial_x F^\varepsilon)^2+\frac{m}{\varepsilon}\partial_x F^\varepsilon\\ &\qquad -\frac{1}{2\varepsilon}\partial_x^2F^\varepsilon-\frac{1}{2}\partial_x F^\varepsilon+\frac{me^{-\varepsilon x}+\partial_x F^\varepsilon}{1-e^{-\varepsilon x}}\end{aligned}\tag{II.3.4.11}$$

and

$$\partial_t\hat{F}^\varepsilon(x,t)=\frac{1}{2\varepsilon}(\partial_x\hat{F}^\varepsilon)^2+\frac{m}{\varepsilon}\partial_x\hat{F}^\varepsilon\,.\tag{II.3.4.12}$$

Remark II.3.4.2. Note that $\hat{F}^1(x,t)$ is the Laplace transform of ρ^0 and that $\hat{F}^\varepsilon(x,t)=\hat{F}^1(\varepsilon x,t)/\varepsilon$. The equation for the scaled Flory solution always becomes (II.3.4.12) after Laplace transform. This means that the solution to this Hamilton-Jacobi equation, after the initial shock, remains smooth at ALL time after gelation time.

Remark II.3.4.3. Observe that as $\varepsilon\to 0$,

$$\partial_x\hat{F}^\varepsilon(x,t)=\partial_x\hat{F}^1(\varepsilon x,t)\to -m_1(t)$$

and

$$\partial_x^2\hat{F}^\varepsilon(x,t)=\varepsilon\partial_x^2\hat{F}^1(\varepsilon x,t)\to 0\,,$$

Letting $G^\varepsilon \stackrel{\text{def}}{=} F^\varepsilon - \hat{F}^\varepsilon$, we have

$$\begin{aligned}
\varepsilon\partial_t G^\varepsilon &= \frac{1}{2}\Big((\partial_x F^\varepsilon)^2 - (\partial_x \hat{F}^\varepsilon)^2\Big) + m\partial_x G^\varepsilon - \frac{1}{2}\partial_x^2 G^\varepsilon \\
&\quad - \frac{1}{2}\partial_x^2 \hat{F}^\varepsilon - \frac{\varepsilon}{2}\partial_x F^\varepsilon + \varepsilon\frac{me^{-\varepsilon x} + \partial_x F^\varepsilon}{1 - e^{-\varepsilon x}} \\
&= \frac{1}{2}(\partial_x G^\varepsilon)^2 + \partial_x F^\varepsilon \partial_x \hat{F}^\varepsilon - (\partial_x \hat{F}^\varepsilon)^2 + m\partial_x G^\varepsilon \\
&\quad - \frac{1}{2}\partial_x^2 G^\varepsilon - \frac{1}{2}\partial_x^2 \hat{F}^\varepsilon - \frac{\varepsilon}{2}\partial_x F^\varepsilon + \varepsilon\frac{me^{-\varepsilon x} + \partial_x F^\varepsilon}{(1 - e^{-\varepsilon x})} \\
&= \frac{1}{2}(\partial_x G^\varepsilon)^2 + (\partial_x \hat{F}^\varepsilon + m)\partial_x G^\varepsilon - \frac{1}{2}\partial_x^2 G^\varepsilon - \frac{1}{2}\partial_x^2 \hat{F}^\varepsilon \\
&\quad + \varepsilon\frac{me^{-\varepsilon x} + \partial_x G^\varepsilon + \partial_x \hat{F}^\varepsilon}{(1 - e^{-\varepsilon x})} - \frac{\varepsilon}{2}\partial_x F^\varepsilon\,.
\end{aligned} \tag{II.3.4.13}$$

This equation is a nonlinear backward parabolic equation, which is far from being understood. We seek to overcome the difficulty of analyzing the equation directly by showing that $\partial_t G^\varepsilon$ is uniformly bounded for each time $t > 0$ and treat the equation as an ODE in space with bounded forcing.

Lemma II.3.4.4. *Let $t > 0$ and $x_0 > 0$, there exists a constant $C = C(x_0) > 0$ such that*

$$\sup_{\varepsilon\in(0,1)} \sup_{[x_0,\infty)} \varepsilon|\partial_t G^\varepsilon(\cdot,t)| \leqslant C\,, \tag{II.3.4.14}$$

and

$$\sup_{\varepsilon\in(0,1)} \sup_{[x_0,\infty)} |\partial_x^2 G^\varepsilon(\cdot,t)| \leqslant C\,. \tag{II.3.4.15}$$

Proof. For the calculations below, the constant $M = M_{x_0}$ will change from line to line but will be independent of ε. By (II.3.4.6) and dominated convergence theorem, for $x > 0$,

$$|\partial_t G^\varepsilon(x,t)| = \Big|\partial_t \sum_{k=1}^{\infty} e^{-k\varepsilon x} h^\varepsilon_{k\varepsilon}(t)\Big| \leqslant C\sum_{k=1}^{\infty} e^{-k\varepsilon x} \leqslant \frac{C}{\varepsilon}\,.$$

Multiply both sides by ε, (II.3.4.14) follows. Furthermore, since every first order derivative terms are uniformly bounded, (II.3.4.15) follows immediately from (II.3.4.13). $\square$

Remark II.3.4.5. As the left hand side of the equation (II.3.4.13) can only be shown to be bounded at the moment, we cannot proceed to the final proof of Conjecture II.3.1.6. If we could improve the bounds in Lemma II.3.4.1, we may be able to show that the left hand side is of order $o(1)$ (as opposed to $O(1)$) as ε vanishes.

A way to show this is to show that $\rho_k^\varepsilon \leqslant C/k^\nu$ for some $\nu > 2$ and $C > 0$ independent of ε. Of course, this requires a detailed study of the rate of decay of the tail of ρ_k^ε, which is a significant challenge. Knowing exactly how the second moment of ρ_k^ε behaves as ε vanishes will also be very useful.

We now show what happens IF it is indeed that for some $x_0 > 0$,

$$\lim_{\varepsilon\to 0} \sup_{[x_0,\infty)} \varepsilon|\partial_t G^\varepsilon(\cdot,t)| = 0\,, \tag{II.3.4.16}$$

PROPOSITION II.3.4.6. *Suppose* (II.3.4.16) *holds. Define* $H^\varepsilon \stackrel{\text{def}}{=} -\partial_x G^\varepsilon = -\partial_x(F^\varepsilon - \hat{F}^\varepsilon)$. *For* $t > 0$, *locally uniformly,*

$$\lim_{\varepsilon\to 0} H^\varepsilon(\cdot,t) = \bar{H}(\cdot,t)$$

where $\bar{H}$ *is the unique bounded solution to the equation*

$$\frac{1}{2}\bar{H}^2 - g(t)\bar{H} + \frac{1}{2}\partial_x\bar{H} + \frac{g(t)-\bar{H}}{x} = 0 \tag{II.3.4.17}$$

that is a completely monotone function.

REMARK II.3.4.7. Of course, when $t \leqslant T_{\text{gel}}$, $\bar{H} = 0$ because $g(t) = 0$ in this case.

REMARK II.3.4.8. We note that the uniqueness might fail in the above proposition if we don't require both boundedness and complete monotonicity.

Before giving the proof of this proposition, we need a few preliminary lemmas concerning the behavior of the solutions of equation (II.3.4.17).

LEMMA II.3.4.9. *Let*

$$H(x,t) = \frac{g(t)}{1 + xg(t)}\,. \tag{II.3.4.18}$$

Then H *solves* (II.3.4.17).

PROOF. This is immediate by a simple calculation. ☐

LEMMA II.3.4.10. *Fix* $t > T_{\text{gel}}$. *There exists a unique bounded solution to equation* (II.3.4.17) *that is a completely monotone function.*

PROOF. We show this by phase plane analysis. Solving the equation

$$\frac{1}{2}u^2 - g(t)u + \frac{g(t)-u}{x} = 0\,,$$

we have that

$$u_1(x,t) = g(t) + \frac{1}{x} - \sqrt{g(t)^2 + \frac{1}{x^2}}\,,$$

and

$$u_2(x,t) = g(t) + \frac{1}{x} + \sqrt{g(t)^2 + \frac{1}{x^2}}\,.$$

On the one hand, it is true that any solution to (II.3.4.15) satisfies

$$\begin{aligned}
\partial_x \bar{H}(x,t) > 0\,, &\quad \bar{H} \in (u_1(x,t), u_2(x,t))\,,\\
\partial_x \bar{H}(x,t) = 0\,, &\quad \bar{H} \in \{u_1(x,t), u_2(x,t)\}\,,\\
\partial_x \bar{H}(x,t) < 0\,, &\quad \bar{H} \notin [u_1(x,t), u_2(x,t)]\,.
\end{aligned}$$

On the other hand, we have

$$\lim_{x\to 0} u_1(x,t) = g(t)\,, \quad \lim_{x\to 0} u_2(x,t) = \infty$$

and

$$\lim_{x\to\infty} u_1(x,t) = 0\,, \quad \lim_{x\to\infty} u_2(x,t) = 2g(t)\,.$$

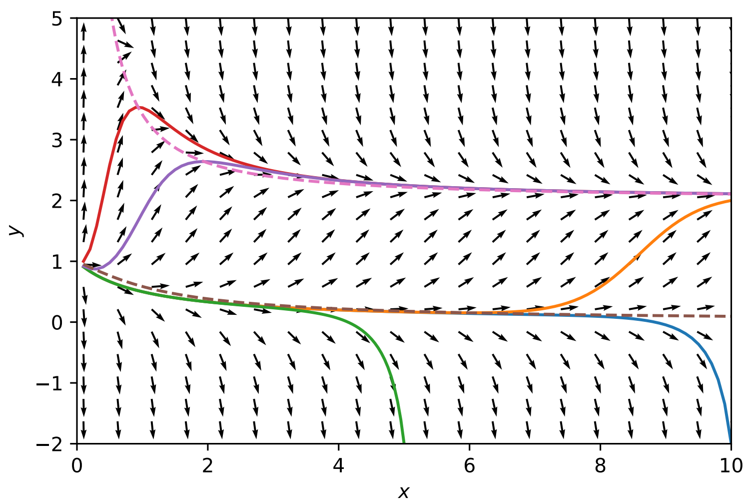

FIGURE II.3.4.1. Solutions to equation (II.3.4.17).

Thus, 0 is an unstable limit point and $2g(t)$ is a stable limit point. Let $\bar{H}$ be given by (II.3.4.18). We note that

$$\bar{H}(1,t) = \frac{g(t)}{(1+g(t))} \quad \text{and} \quad \lim_{x\to\infty} \bar{H}(x,t) = 0\,.$$

Therefore, if $\tilde{H}$ is another solution of equation (II.3.4.17), there are two cases:

(I) $$\lim_{x\to x_0} H(x,t) = -\infty \quad \text{if} \quad H(1,t) < \frac{g(t)}{1+g(t)}\,,$$

$$\text{(II)} \qquad \lim_{x\to\infty} H(x,t) = 2g(t) \quad \text{if} \quad H(1,t) > \frac{g(t)}{1+g(t)}\,,$$

where $x_0 \in (0,\infty]$. In case (II), because $\lim_{x\to 0} u_t^2(x) = \infty$, any bounded solution must be increasing initially near $x = 0$ and, therefore, cannot be a completely monotone function. Thus, the only bounded solution that is a completely monotone function is $\bar{H}$, proving Lemma II.3.4.10. $\square$

Proof of Proposition II.3.4.6. Recall that $H^\varepsilon \stackrel{\text{def}}{=} -\partial_x G^\varepsilon$ and G^ε satisfies (II.3.4.13). By equation (II.3.4.16) and Arzelà-Ascoli theorem, there exists a function $\bar{H}(\cdot,t)$ such that, up to a subsequence, $H^\varepsilon(\cdot,t) \to \bar{H}(\cdot,t)$ locally uniformly. As $|H^\varepsilon| \leqslant m$, $|\bar{H}| \leqslant m$ as well.

Lastly, as $\partial_x \hat{F}^\varepsilon(x,t) \to -m + g(t) < 0$ locally uniformly, $\bar{H}$ is a completely monotone function as it is the locally uniform limit of $\{-\partial_x F^\varepsilon - m + g(t)\}$, a sequence of completely monotone functions. By the uniqueness of the completely monotone solution to (II.3.4.17), the convergence above is true for the full sequence $\{H^\varepsilon\}$. The proof is finished. $\square$

Finally, we will show that if (II.3.4.16) is true then Conjecture II.3.1.6 is true.

A conditional proof of Conjecture II.3.1.6. Suppose it is true that (II.3.4.16) holds. Let $\bar{\ell}(s,t) = s\bar{c}(s,t) = e^{-s/g(t)}$. Denote by $L(x,t) = \mathcal{L}(\bar{\ell}(\cdot,t))(x)$ the Laplace transform of $\bar{\ell}$. A simple calculation shows that

$$L(x,t) = \frac{g(t)}{1+xg(t)}\,.$$

The limit (II.3.4.1) follows immediately from Proposition II.3.4.6 and the continuity theorem for the Laplace transform (see for example [Fel71]), which proves Conjecture II.3.1.6. $\square$

Bibliography

[AB79] Michael Aizenman and Thor A. Bak. “Convergence to Equilibrium in a System of Reacting Polymers”. In: *Communications in Mathematical Physics* 65.3 (1979), pp. 203–230. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103904874.

[AK18] Tabish Alam and Man-Hoe Kim. “A comprehensive review on single phase heat transfer enhancement techniques in heat exchanger applications”. In: *Renewable and Sustainable Energy Reviews* 81 (2018), pp. 813–839. DOI: 10.1016/j.rser.2017.08.060.

[Ald99] David J. Aldous. “Deterministic and Stochastic Models for Coalescence (Aggregation and Coagulation): A Review of the Mean-Field Theory for Probabilists”. In: *Bernoulli. Official Journal of the Bernoulli Society for Mathematical Statistics and Probability* 5.1 (1999), pp. 3–48. ISSN: 1350-7265. DOI: 10.2307/3318611.

[All92] Grégoire Allaire. “Homogenization and two-scale convergence”. In: *SIAM J. Math. Anal.* 23.6 (1992), pp. 1482–1518. ISSN: 0036-1410. DOI: 10.1137/0523084.

[BC90] J. M. Ball and J. Carr. “The Discrete Coagulation-Fragmentation Equations: Existence, Uniqueness, and Density Conservation”. In: *Journal of Statistical Physics* 61.1-2 (1990), pp. 203–234. ISSN: 0022-4715. DOI: 10.1007/BF01013961.

[BC97] Martino Bardi and Italo Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. With appendices by Maurizio Falcone and Pierpaolo Soravia. Birkhäuser Boston, Inc., Boston, MA, 1997, pp. xviii+570. ISBN: 0-8176-3640-4. DOI: 10.1007/978-0-8176-4755-1.

[Ber06] Jean Bertoin. *Random Fragmentation and Coagulation Processes*. Vol. 102. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2006. viii+280. ISBN: 978-0-521-86728-3 0-521-86728-2. DOI: 10.1017/CBO9780511617768.

[BK11] E. Ben-Naim and P. L. Krapivsky. “Kinetics of Ring Formation”. In: *Physical Review E* 83.6 (June 2011), p. 061102. DOI: 10.1103/PhysRevE.83.061102.

[BLL19] Jacek Banasiak, Wilson Lamb, and Philippe Laurencot. *Analytic Methods for Coagulation-Fragmentation Models*. Vol. 1&2. CRC Press, 2019.

[BLP78] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic analysis for periodic structures*. Vol. 5. Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam-New York, 1978, pp. xxiv+700. ISBN: 0-444-85172-0.

[BT45] P. J. Blatz and A. V. Tobolsky. "Note on the Kinetics of Systems Manifesting Simultaneous Polymerization-Depolymerization Phenomena". In: *The Journal of Physical Chemistry* 49.2 (1945), pp. 77–80. DOI: 10.1021/j150440a004.

[Cal18] Jeff Calder. "Lecture notes on viscosity solutions". In: (2018). URL: http://www-users.math.umn.edu/~jwcalder/8590F18/viscosity_solutions.pdf.

[Cañ07] José A. Cañizo. "Convergence to Equilibrium for the Discrete Coagulation-Fragmentation Equations with Detailed Balance". In: *Journal of Statistical Physics* 129.1 (2007), pp. 1–26. ISSN: 0022-4715. DOI: 10.1007/s10955-007-9373-2.

[Car92] J. Carr. "Asymptotic Behaviour of Solutions to the Coagulation-Fragmentation Equations. I. The Strong Fragmentation Case". In: *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics* 121.3-4 (1992), pp. 231–244. ISSN: 0308-2105. DOI: 10.1017/S0308210500027888.

[CEL84] M. G. Crandall, L. C. Evans, and P.-L. Lions. "Some Properties of Viscosity Solutions of Hamilton-Jacobi Equations". In: *Transactions of the American Mathematical Society* 282.2 (1984), pp. 487–502. ISSN: 0002-9947. DOI: 10.2307/1999247.

[CL83] Michael G. Crandall and Pierre-Louis Lions. "Viscosity Solutions of Hamilton-Jacobi Equations". In: *Transactions of the American Mathematical Society* 277.1 (1983), pp. 1–42. ISSN: 0002-9947. DOI: 10.2307/1999343.

[Cos95] Fernando Pestana da Costa. "Existence and Uniqueness of Density Conserving Solutions to the Coagulation-Fragmentation Equations with Strong Fragmentation". In: *Journal of mathematical analysis and applications* (1995), pp. 892–914.

[Cra97] Michael G. Crandall. "Viscosity solutions: a primer". In: *Viscosity solutions and applications (Montecatini Terme, 1995)*. Vol. 1660. Lecture Notes in Math. Springer, Berlin, 1997, pp. 1–43. DOI: 10.1007/BFb0094294.

[DC96] Charles R. Doering and Peter Constantin. "Variational bounds on energy dissipation in incompressible flows. III. Convection". In: *Physical Review E* 53.6 (June 1996), pp. 5957–5981. DOI: 10.1103/physreve.53.5957.

[dCos15] F. P. da Costa. "Mathematical Aspects of Coagulation-Fragmentation Equations". In: *Mathematics of Energy and Climate Change*. Vol. 2. CIM Ser. Math. Sci. Springer, Cham, 2015, pp. 83–162.

[DH98] P. Daskalopoulos and R. Hamilton. "Regularity of the Free Boundary for the Porous Medium Equation". In: *Journal of the American Mathematical Society* 11.4 (1998), pp. 899–965. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-98-00277-X.

[DLP17] Pierre Degond, Jian-Guo Liu, and Robert L. Pego. "Coagulation-fragmentation model for animal group-size statistics". In: *Journal of Nonlinear Science* 27.2 (2017), pp. 379–424. ISSN: 0938-8974. DOI: 10.1007/s00332-016-9336-3.

[DOR06] Charles R. Doering, Felix Otto, and Maria G. Reznikoff. "Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh-Bénard convection". In: *Journal of Fluid Mechanics* 560 (2006), pp. 229–241. ISSN: 0022-1120. DOI: 10.1017/S0022112006000097.

[DT19a] Charles R. Doering and Ian Tobasco. "On the optimal design of wall-to-wall heat transport". In: *Communications on Pure and Applied Mathematics* 72.11 (2019), pp. 2385–2448. ISSN: 0010-3640. DOI: 10.1002/cpa.21832.

[DT19b] Charles R. Doering and Ian Tobasco. "On the optimal design of wall-to-wall heat transport". In: *Communications on Pure and Applied Mathematics* 72.11 (2019), pp. 2385–2448. ISSN: 0010-3640. DOI: 10.1002/cpa.21832.

[EMP02] M. Escobedo, S. Mischler, and B. Perthame. "Gelation in Coagulation and Fragmentation Models". In: *Communications in Mathematical Physics* 231.1 (2002), pp. 157–188. ISSN: 0010-3616. DOI: 10.1007/s00220-002-0680-9.

[Esc+03] M. Escobedo et al. "Gelation and Mass Conservation in Coagulation-Fragmentation Models". In: *Journal of Differential Equations* 195.1 (2003), pp. 143–174. ISSN: 0022-0396. DOI: 10.1016/S0022-0396(03)00134-7.

[Eva10] Lawrence C. Evans. *Partial Differential Equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019.

[Eva90] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*. Vol. 74. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990, pp. viii+80. ISBN: 0-8218-0724-2. DOI: 10.1090/cbms/074.

[Fel71] W. Feller. *An Introduction to Probability Theory and Its Applications. Vol. II.* Second Edition. New York: John Wiley & Sons Inc., 1971. xxiv+669.

[FM04] Nicolas Fournier and Stéphane Mischler. "Exponential trend to equilibrium for discrete coagulation equations with strong fragmentation and without a balance condition". In: *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* 460.2049 (2004), pp. 2477–2486. ISSN: 1364-5021. DOI: 10.1098/rspa.2004.1294.

[Fou09] Jean Baptiste Joseph Fourier. *Théorie analytique de la chaleur*. Cambridge Library Collection. Reprint of the 1822 original, Previously published by Éditions Jacques Gabay, Paris, 1988 [MR1414430]. Cambridge University Press, Cambridge, 2009, pp. ii+xxii+643. ISBN: 978-1-108-00180-9. DOI: 10.1017/CBO9780511693229.

[FP13] Paul M. N. Feehan and Camelia A. Pop. "A Schauder Approach to Degenerate-Parabolic Partial Differential Equations with Unbounded Coefficients". In: *Journal of Differential Equations* 254.12 (2013), pp. 4401–4445. ISSN: 0022-0396. DOI: 10.1016/j.jde.2013.03.006.

[FW12] Mark I. Freidlin and Alexander D. Wentzell. *Random perturbations of dynamical systems*. Third. Vol. 260. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the 1979 Russian original by Joseph Szücs. Springer, Heidelberg, 2012, pp. xxviii+458. ISBN: 978-3-642-25846-6. DOI: 10.1007/978-3-642-25847-3.

[GHZ11] Martin Glässl, Markus Hilt, and Walter Zimmermann. "Convection in nanofluids with a particle-concentration-dependent thermal conductivity". In: *Physical Review E* 83.4 (2011), p. 046315. DOI: 10.1103/physreve.83.046315.

[GMT16] Yoshikazu Giga, Hiroyoshi Mitake, and Hung V. Tran. "On Asymptotic Speed of Solutions to Level-Set Mean Curvature Flow Equations with Driving and Source Terms". In: *SIAM J. Math. Anal.* 48(5) (2016), pp. 3515–3546. DOI: 10.1137/15M1052755.

[ILP18] G. Iyer, N. Leger, and R. L. Pego. "Coagulation and Universal Scaling Limits for Critical Galton-Watson Processes". In: *Advances in Applied Probability* (2018).

[IS12] Hitoshi Ishii and Panagiotis E. Souganidis. "A PDE approach to small stochastic perturbations of Hamiltonian flows". In: *J. Differential Equations* 252.2 (2012), pp. 1748–1775. ISSN: 0022-0396. DOI: 10.1016/j.jde.2011.08.036.

[IV21] Gautam Iyer and Truong-Son Van. "Bounds on the heat transfer rate via passive advection". In: (Jan. 25, 2021). arXiv: 2101.10287 [math.AP].

[Kad01] Leo P. Kadanoff. "Turbulent Heat Flow: Structures and Scaling". In: *Physics Today* 54.8 (2001), pp. 34–39. DOI: 10.1063/1.1404847.

[KJ03] Jun Korenaga and Thomas H. Jordan. "Physics of multiscale convection in Earth's mantle: Onset of sublithospheric convection". In: *Journal of Geophysical Research: Solid Earth* 108.B7 (2003). DOI: 10.1029/2002jb001760.

[Koc98] H. Koch. *Non-Euclidean Singular Integrals and the Porous Medium Equation*. Verlag nicht ermittelbar, 1998. URL: https://books.google.com/books?id=nNjnPgAACAAJ.

[Kry96] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*. Vol. 12. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1996. xii+164. ISBN: 0-8218-0569-X. DOI: 10.1090/gsm/012.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. Second. Vol. 113. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991. xxiv+470. ISBN: 0-387-97655-8. DOI: 10.1007/978-1-4612-0949-2.

[Kum18] Taiga Kumagai. "A perturbation problem involving singular perturbations of domains for Hamilton-Jacobi equations". In: *Funkcial. Ekvac.* 61.3 (2018), pp. 377–427. ISSN: 0532-8721.

[Kus84] Harold J. Kushner. *Approximation and weak convergence methods for random processes, with applications to stochastic systems theory.* MIT Press Series in Signal Processing, Optimization, and Control, 6. MIT Press, Cambridge, MA, 1984, pp. xvii+269. ISBN: 0-262-11090-3.

[Lau19a] Philippe Laurençot. "Mass-Conserving Solutions to Coagulation-Fragmentation Equations with Balanced Growth". In: (2019). arXiv: 1901.08313 [math.AP].

[Lau19b] Philippe Laurençot. "Stationary Solutions to Coagulation-Fragmentation Equations". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* (2019). ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2019.06.003. URL: http://www.sciencedirect.com/science/article/pii/S0294144919300617.

[Ley03] F. Leyvraz. "Scaling Theory and Exactly Solved Models in the Kinetics of Irreversible Aggregation". In: *Physics Reports* 383.2-3 (Aug. 2003), pp. 95–212. DOI: 10.1016/S0370-1573(03)00241-2.

[Lie96] Gary M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439. ISBN: 981-02-2883-X. DOI: 10.1142/3302.

[Lio84] P.-L. Lions. "The concentration-compactness principle in the calculus of variations. The locally compact case. I". In: *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire* 1.2 (1984), pp. 109–145. ISSN: 0294-1449.

[LL20] John H. Lienhard IV and John H. Lienhard V. *A Heat Transfer Textbook*. Dover Publications Inc., Jan. 31, 2020. 784 pp. ISBN: 0486837351. URL: https://www.ebook.de/de/product/38396259/john_lienhard_a_heat_transfer_textbook.html.

[LNP19] Jian-Guo Liu, B. Niethammer, and Robert L. Pego. "Self-Similar Spreading in a Merging-Splitting Model of Animal Group Size". In: *Journal of Statistical Physics* 175.6 (2019), pp. 1311–1330. ISSN: 0022-4715. DOI: 10.1007/s10955-019-02280-w.

[LSU68] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*. Translated from the

Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968. xi+648.

[Mar+18a] Florence Marcotte et al. "Optimal Heat Transfer and Optimal Exit Times". In: *SIAM Journal on Applied Mathematics* 78.1 (2018), pp. 591–608. ISSN: 0036-1399. DOI: 10.1137/17M1150220.

[Mar+18b] Florence Marcotte et al. "Optimal heat transfer and optimal exit times". In: *SIAM J. Appl. Math.* 78.1 (2018), pp. 591–608. ISSN: 0036-1399. DOI: 10.1137/17M1150220.

[McL62] J. B. McLeod. "On an Infinite Set of Non-Linear Differential Equations". In: *The Quarterly Journal of Mathematics. Oxford. Second Series* 13 (1962), pp. 119–128. ISSN: 0033-5606. DOI: 10.1093/qmath/13.1.119.

[Mel57] Z. A. Melzak. "A Scalar Transport Equation". In: *Transactions of the American Mathematical Society* 85 (1957), pp. 547–560. ISSN: 0002-9947. DOI: 10.2307/1992943.

[MP04] Govind Menon and Robert L. Pego. "Approach to Self-Similarity in Smoluchowski's Coagulation Equations". In: *Communications on Pure and Applied Mathematics* 57.9 (2004), pp. 1197–1232. ISSN: 0010-3640. DOI: 10.1002/cpa.3048.

[MP06] Govind Menon and Robert L. Pego. "Dynamical scaling in Smoluchowski's coagulation equations: uniform convergence". In: *SIAM Rev.* 48.4 (2006), pp. 745–768. ISSN: 0036-1445. DOI: 10.1137/060662496.

[MP08] Govind Menon and Robert L. Pego. "The Scaling Attractor and Ultimate Dynamics for Smoluchowski's Coagulation Equations". In: *Journal of Nonlinear Science* 18.2 (2008), pp. 143–190. ISSN: 0938-8974. DOI: 10.1007/s00332-007-9007-5.

[MTV21] Hiroyoshi Mitake, Hung V. Tran, and Truong-Son Van. "Large time behavior for a Hamilton-Jacobi equation in a critical Coagulation-Fragmentation model". In: *Communications in Mathematical Sciences* (2021). DOI: 10.4310/CMS.2021.v19.n2.a8.

[Ngu89] Gabriel Nguetseng. "A general convergence result for a functional related to the theory of homogenization". In: *SIAM J. Math. Anal.* 20.3 (1989), pp. 608–623. ISSN: 0036-1410. DOI: 10.1137/0520043.

[Nor99] James R. Norris. "Smoluchowski's Coagulation Equation: Uniqueness, Nonuniqueness and a Hydrodynamic Limit for the Stochastic Coalescent". In: *The Annals of Applied Probability* 9.1 (1999), pp. 78–109. ISSN: 1050-5164. DOI: 10.1214/aoap/1029962598.

[NV13] B. Niethammer and J. J. L. Velázquez. "Self-Similar Solutions with Fat Tails for Smoluchowski's Coagulation Equation with Locally Bounded Kernels". In: *Communications in Mathematical Physics* 318.2 (2013), pp. 505–532. ISSN: 0010-3616. DOI: 10.1007/s00220-012-1553-5.

[Øks03a] Bernt Øksendal. *Stochastic Differential Equations*. Sixth. Universitext. Springer-Verlag, Berlin, 2003. xxiv+360. ISBN: 3-540-04758-1. DOI: 10.1007/978-3-642-14394-6.

[Øks03b] Bernt Øksendal. *Stochastic differential equations*. Sixth. Universitext. An introduction with applications. Springer-Verlag, Berlin, 2003, pp. xxiv+360. ISBN: 3-540-04758-1. DOI: 10.1007/978-3-642-14394-6.

[Ost17] R. Ostilla-Mónico. "Mixed thermal conditions in convection: how do continents affect the mantle's circulation?" In: *Journal of Fluid Mechanics* 822 (2017), pp. 1–4. DOI: 10.1017/jfm.2017.247.

[PS08] Grigorios A. Pavliotis and Andrew M. Stuart. *Multiscale methods – Averaging and homogenization*. Vol. 53. Texts in Applied Mathematics. New York: Springer, 2008, pp. xviii+307. ISBN: 978-0-387-73828-4.

[PV21] Robert L. Pego and Truong-Son Van. "On the dynamics of a singularly perturbed multiplicative-additivecoagulation-fragmentation equation". In: *In Preparation* (2021).

[QM02] Weilin Qu and Issam Mudawar. "Analysis of three-dimensional heat transfer in micro-channel heat sinks". In: *International Journal of Heat and Mass Transfer* 45.19 (2002), pp. 3973–3985. DOI: 10.1016/s0017-9310(02)00101-1.

[Ray16] Lord Rayleigh. "LIX. On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 32.192 (1916), pp. 529–546. DOI: 10.1080/14786441608635602.

[SG88] T. H. Solomon and J. P. Gollub. "Passive transport in steady Rayleigh-Bénard convection". In: *The Physics of Fluids* 31.6 (1988), pp. 1372–1379. DOI: 10.1063/1.866729.

[She+19] M. Sheikholeslami et al. "Heat transfer simulation of heat storage unit with nanoparticles and fins through a heat exchanger". In: *International Journal of Heat and Mass Transfer* 135 (2019), pp. 470–478. DOI: 10.1016/j.ijheatmasstransfer.2019.02.003.

[Smo16a] M. V. Smoluchowski. "Drei Vortrage Uber Diffusion, Brownsche Bewegung Und Koagulation von Kolloidteilchen". In: *Zeitschrift fur Physik* 17 (1916), pp. 557–585.

[Smo16b] Marian von Smoluchowski. "Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen". In: *Physik. Z.* 17 (1916), pp. 557–585.

[SSV12] René L. Schilling, Renming Song, and Zoran Vondraček. *Bernstein Functions*. Second. Vol. 37. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 2012. xiv+410. ISBN: 978-3-11-025229-3 978-3-11-026933-8. DOI: 10.1515/9783110269338.

[Str08a] Walter A. Strauss. *Partial differential equations*. Second. An introduction. John Wiley & Sons, Ltd., Chichester, 2008, pp. x+454. ISBN: 978-0-470-05456-7.

[Str08b] Michael Struwe. *Variational methods*. Fourth. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Applications to nonlinear partial differential equations and Hamiltonian systems. Springer-Verlag, Berlin, 2008, pp. xx+302. ISBN: 978-3-540-74012-4.

[Tra21] Hung V. Tran. *Hamilton–Jacobi equations: theory and applications*. AMS Graduate Studies in Mathematics, 2021.

[TS02] Masahisa Tabata and Atsushi Suzuki. "Mathematical modeling and numerical simulation of Earth's mantle convection". In: *Mathematical modeling and numerical simulation in continuum mechanics (Yamaguchi, 2000)*. Vol. 19. Lect. Notes Comput. Sci. Eng. Springer, Berlin, 2002, pp. 219–231.

[TV21] Hung V. Tran and Truong-Son Van. "Coagulation-Fragmentation equations with multiplicative coagulation kernel and constant fragmentation kernel". In: *Communications on Pure and Applied Mathematics* (2021). DOI: 10.1002/cpa.21979.

[VP14] Sylvain Vallaghé and Anthony T. Patera. "The static condensation reduced basis element method for a mixed-mean conjugate heat exchanger model". In: *SIAM Journal on Scientific Computing* 36.3 (2014), B294–B320. ISSN: 1064-8275. DOI: 10.1137/120887709.

[VZ89] R. D. Vigil and R. M. Ziff. "On the Stability of Coagulation–Fragmentation Population Balances". In: *Journal of Colloid and Interface Science* 133 (Nov. 1989), pp. 257–264. DOI: 10.1016/0021-9797(89)90300-7.

[Wan+18] Yabo Wang et al. "Energy saving potential of using heat pipes for CPU cooling". In: *Applied Thermal Engineering* 143 (2018), pp. 630–638. DOI: 10.1016/j.applthermaleng.2018.07.132.

[WBZ92] Arel Weisberg, Haim H. Bau, and J.N. Zemel. "Analysis of microchannels for integrated cooling". In: *International Journal of Heat and Mass Transfer* 35.10 (1992), pp. 2465–2474. DOI: 10.1016/0017-9310(92)90089-b.

[YVL15] Yantao Yang, Roberto Verzicco, and Detlef Lohse. "From convection rolls to finger convection in double-diffusive turbulence". In: *Proceedings of the National Academy of Sciences* 113.1 (2015), pp. 69–73. DOI: 10.1073/pnas.1518040113.