Dissipation Enhancement by Mixing

Yuanyuan Feng

A DISSERTATION

in

Mathematical Sciences

Presented to the Faculties of Carnegie Mellon University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

May 2019

Gautam Iyer, advisor

Dissertation Committee:
Gautam Iyer, Robert Pego, Dejan Slepčev, Dehua Wang.

Acknowledgments

First I would like to thank my advisor Gautam Iyer for introducing me to this exciting field and for his time and great patience to work with me. Thank Gautam not only for his nice research-wise suggestions when running into difficulties, but also for his great psychological support. He helps me build confidence in mathematics study and always makes me feel that doing math is a very enjoyable thing. He has always make him available for help and advice. He shares his broad mathematical views with me and teaches me how to survive in academia. His guidance and support has been amazing, from which I would benefit for lifetime.

Thank all the professors who have taught my courses at CMU. Thank you for leading me to different worlds of mathematics and aiding me to build the foundation of my own castle in mathematics. Especially, I would like to thank Giovanni Leoni, who has taught me five courses and definitely gets me well shaped for my later research. And thank Bob Pego, Dejan Slepčev, Ian Tice for giving the interesting topic classes. Those topic classes have broaden my view and inspired me to explore more areas. Thank Bill Hrusa for taking me to do the reading course and giving me so many helps through all my graduate period. Especially thank him for giving me so much helpful advice on my files for postdocotoral application.

Thank all my collaborators. Discussion math with you is very exciting and beneficial. I have learned a lot through working with you all.

Thank all my friends, both mathematicians and non-mathematicians. For my mathematician friends, thank you for discussion math with me, sharing feelings when I got stuck in research, and giving helpful advice. Especially, thank Deborah Brandon, who is a very helpful elder for me. She helped me improve my spoken English, shared her own experience on becoming a mathematician and took me to explore a more interesting life at Pittsburgh. For my non-mathematician friends, thank you for taking me into another funny world every time when I stay away from math for a while. I feel very grateful for having you all on my way of growing into a mathematician.

Thank my husband Lei Li, who is also a young mathematician. Thank him for being my best companion, both in math and life.

Thank my parents for always being my great listeners and supporters. Their unconditional love makes me strong and never be afraid of failure.

ABSTRACT

In this thesis, we quantitatively study the interaction between diffusion and mixing in both the continuous, and discrete time setting. In discrete time, we consider a mixing dynamical system interposed with diffusion. In continuous time, we consider the advection diffusion equation where the advecting vector field is assumed to be sufficiently mixing. The main result is to estimate the dissipation time and energy decay based on an assumption quantifying the mixing rate.

The thesis is mainly based on the paper [FI19].

Contents

Chapter 1

Introduction

Diffusion and mixing are two fundamental phenomena that arise in a wide variety of applications ranging from micro-fluids to meteorology, and even cosmology. In incompressible fluids, stirring induces mixing by filamentation and facilitates the formation of small scales. Diffusion, on the other hand, efficiently damps small scales and the balance between these two phenomena is the main subject of our investigation. Specifically, our aim is to quantify the interaction between diffusion and mixing in a manner that often arises in the context of fluids [DT06, CKRZ08, LTD11, Thi12].

In the absence of diffusion, the mixing of tracer particles passively advected by an incompressible flow has been extensively studied. Several authors [MMP05, LTD11, Thi12] measured mixing using *multi-scale* norms and studied how efficiently incompressible flows can mix (see for instance [Bre03, LLN+12, IKX14, ACM16, YZ17] and references therein). In this scenario, however, there is no apriori limit to the resolution attainable via mixing.

In contrast, in the presence of diffusion, the effects of mixing may be enhanced, balanced, or even counteracted by diffusion (see for instance [FP94, TC03, FNW04, CKRZ08, INRZ10, KX15, MDTY18, MD18]). In this thesis we quantify this interaction by studying the energy dissipation rate. Explicitly, we study the advection diffusion equation, and establish several explicit relations between the *mixing rate* of the drift and the energy dissipation rate.

In this chapter, we begin by introducing the notion of mixing from the dynamical system perspective in Section 1.1. Next, in Section 1.2 we describe mixing rates in a manner that is well suited to the study of the advection diffusion equation. In Section 1.3, we introduce the notion of *dissipation time* in the context of pulsed diffusions and the advection diffusion equation.

Finally in Section 1.4, we conclude this chapter with the statement of the main results of this thesis.

1.1 Ergodicity and Mixing

Many physical phenomena can be modeled as dynamical systems, and ergodic theory is the study of the long time behavior of these dynamical systems. The abstract framework in ergodic theory starts with a probability space $(X, \mathcal{B}, \mu)$, and a measure preserving transformation $T\colon X \to X$ that represents the time dynamics.

We recall that T is ergodic, if the only invariant sets are either null or co-null (i.e. if for any $A \in \mathcal{B}$ we have $T^{-1}(A) = A$, then $\mu(A) \in \{0, 1\}$). If T is ergodic, then the Birkhoff ergodic theorem (see for instance [EFHN15, Corollary 11.2]) guarantees that for any $f \in L^1(X)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int_X f \, d\mu \,, \quad \text{for almost every } x \in X \,.$$

Roughly speaking, this means that if T is ergodic then the phase-space averages and time averages are equal.

Given any two sets $A, B \in \mathcal{B}$, we apply the Birkhoff ergodic theorem to the function $f = \mathbf{1}_A$, and multiply by $\mathbf{1}_B$. This gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(T^{-j}A \cap B) = \mu(A)\mu(B) \quad \text{for any } A, B \in \mathcal{B} \,. \tag{1.1.1}$$

From probabilistic point of view, we recall that two events A, B are independent if $\mu(A \cap B) = \mu(A)\mu(B)$. Thus the ergodicity of T tells us that, for any pair of events $A, B \in \mathcal{B}$, the events $T^{-n}A$, B become approximately independent as $n \to \infty$. The notions of *strong mixing* and *weak mixing*, defined below, impose stronger requirements on the sense in which $T^{-n}A, B$ become independent as $n \to \infty$.

Definition 1.1.1. Let T be a measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$.

(1) We say that T is *weakly mixing*, if for every $A, B \in \mathcal{B}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T^{-j}A \cap B) - \mu(A)\mu(B)| = 0 \,.$$

(2) We say that T is *strongly mixing*, if for every $A, B \in \mathcal{B}$, we have

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)\,.$$

Clearly strong-mixing implies weak-mixing, which in turn implies ergodicity. The converse implications, however, are both false. Roughly speaking, strongly mixing says that for every Borel set A, successive iterations of the map T will stretch and fold it over X so that it eventually the fraction of *every* fixed region $B \subseteq X$ occupied by iterates of A will approach $\mu(A)$. Figure 1.1 shows an example of the dynamics of successive iterations of a flow on the Torus that is conjectured to be strongly mixing [Pie94]. For a comprehensive review and more examples we refer the reader to [KH95, SOW06, EFHN15].

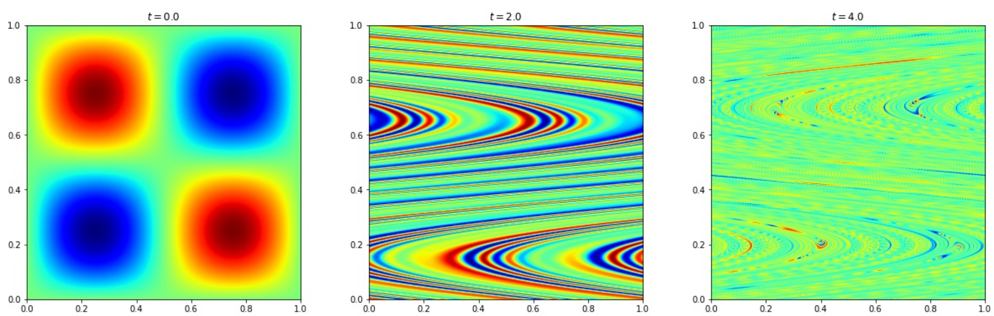

Figure 1.1: An example of mixing on the torus.

In general checking a particular dynamical systems is mixing is not an easy task. While there are several equivalent characterizations of ergodicity and mixing, none of these criterion are easy to check. The best known equivalent characterizations involve spectral conditions, which we now describe. In order to describe these conditions, we first reformulate our setup functionally.

Note that the dynamical system $T\colon X \to X$ induces an isometry U on $L^2(X, \mathcal{B}, \mu)$ defined by composition:

$$Uf \stackrel{\text{def}}{=} f \circ T \quad \text{for all } f \in L^2(X, \mathcal{B}, \mu)\,.$$

The operator U is known as the *Koopman operator* associated with T. Recall $L^2(X, \mathcal{B}, \mu)$, hereafter abbreviated to L^2, denotes the space of complex-valued square-integrable functions on X with the inner-product

$$\langle f, g\rangle = \int_X f\bar{g}\, d\mu\,.$$

The fact that T is measure preserving is equivalent to the operator U being an isometry. If additionally T is invertible, then U is unitary.

As we will see below, the ergodicity and mixing of the dynamical system T is intrinsically related to spectral properties of U.

Proposition 1.1.2. *Let $T\colon X \to X$ be a measure preserving transformation.*

1. *The map T is* ergodic *if and only if* 1 *is a simple eigenvalue of* U*.*

2. *The map T is* weakly mixing *if and only if the only eigenvalue of* U *is* 1*, and it is a simple eigenvalue.*

The first part of this proposition is easily verified. Indeed, the definition of ergodicity is equivalent to the fact that $Tf = f$ if and only if f is a constant. This immediately yields the first part of Proposition 1.1.2. The proof of the second part is more involved, and we refer the reader to...

Invertible strongly mixing operators can also be characterized spectrally by the property that its maximal spectral type consists of only Rajchman measures. Since the precise definitions of this would involve too long a digression, we refer the reader to [EFHN15, Chapter 18.4.2] instead.

We conclude this section with a few well known examples of ergodic and mixing maps.

Example 1.1.3 (Irrational shift)*.* Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and define $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by

$$T(x) = x + \alpha \mod 1\,.$$

It is well known that T is ergodic, but not mixing, with respect to the Lebesgue measure (see for instance [EFHN15, Chapter 9.2]).

Example 1.1.4 (Uniformly expanding maps)*.* Define $\varphi\colon \mathbb{T}^1 \to \mathbb{T}^1$ by

$$\varphi(x) = mx \mod 1\,,$$

for any $m \in \mathbb{Z}$ with $m \geqslant 2$. It is well known that φ strongly mixing with respect to the Lebesgue measure (see for instance [KH95, Proposition 4.2.11]). We will revisit this example in Chapter 1.2.1, and prove a stronger result showing that uniformly expanding maps are in fact *exponentially mixing*.

Example 1.1.5*.* Consider the Bernoulli shift $U : \ell^2 \to \ell^2$ defined by

$$U(x)_n = x_{n-1} \text{ for } n \geqslant 2\ \ ,$$

$$U(x)_1 = 0\,.$$

It is easy to see that the Bernoulli shift U is strongly mixing, and we refer the reader to [EFHN15, Chapter 6.2] for the proof.

Example 1.1.6. The *Bakers map* and *cat map* are two other well known examples of mixing maps, and are described in the next section.

1.2 Mixing Rates

In order to study rates of mixing, one needs to impose additional structure on the underlying space. Instead of working on an abstract measure space, we now work a d-dimensional Riemannian manifold M. For simplicity we assume the metric is normalized so that the total volume of M is 1. In this context, a volume preserving diffeomorphism $\varphi\colon M \to M$ is said to be *mixing* (or *strongly mixing*) if for every pair of Borel sets $A, B \subseteq M$, we have

$$\lim_{n\to\infty} \operatorname{vol}(\varphi^{-n}(A) \cap B) = \operatorname{vol}(A)\operatorname{vol}(B)\,. \tag{1.2.1}$$

Approximating by simple functions we see that (1.2.1) immediately implies that for any $f, g \in L^2_0$, we have[1]

$$\lim_{n\to\infty} \langle U^n f, g\rangle = 0\,.$$

Thus, one can quantify the *mixing rate* by requiring the correlations $\langle U^n f, g\rangle$ to decay at a particular rate. Since these are linear in f, g, a natural first attempt is to require

$$\Big|\langle U^n f, g\rangle\Big| \leqslant h(n)\|f\|\,\|g\|\,, \tag{1.2.2}$$

where $\|f\|$, $\|g\|$ denote the L^2 norms of f and g, respectively, and $h(n)$ is some decreasing sequence that vanishes at infinity. This, however, is impossible. Indeed using duality, equation (1.2.2) immediately implies

$$\|U^n f\| \leqslant h(n)\|f\| \xrightarrow{n\to\infty} 0\,. \tag{1.2.3}$$

Of course, U is a unitary operator and hence we must also have $\|U^n f\| = \|f\|$, which is in direct contradiction to (1.2.3).

[1]Recall L^2_0 is the set of all mean zero square integrable functions, and $U\colon L^2_0 \to L^2_0$ is the Koopman operator defined by $Uf = f \circ \varphi$.

To circumvent this difficulty, one uses stronger norms of f and g on the right of (1.2.2). The traditional choice in the dynamical systems literature is to use Hölder norms. However, following Fannjiang et. al. [FW03, FNW04, FNW06], we use Sobolev norms instead, as it is more convenient for our purposes.

Definition 1.2.1. Let $h\colon \mathbb{N} \to (0, \infty)$ be a decreasing function that vanishes at infinity, and $\alpha, \beta > 0$. We say that φ is *strongly* α*,* β *mixing with rate function* h if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ the associated Koopman operator U satisfies

$$\Big|\langle U^n f, g\rangle\Big| \leqslant h(n) \|f\|_\alpha \|g\|_\beta \,. \tag{1.2.4}$$

Here $\dot{H}^\alpha = \dot{H}^\alpha(M)$ is the homogeneous Sobolev space of order α, and $\|\cdot\|_\alpha$ denotes the norm in $\dot{H}^\alpha$.

Remark 1.2.2. It is easy to see that if U is unitary, then one must have both $\alpha > 0$ and $\beta > 0$. If, however, U is only an isometry (and not necessarily invertible) it is possible to find examples where either $\alpha > 0$, or $\beta > 0$. The uniformly expanding map (discussed below) is one such example.

Remark 1.2.3. When U is the Koopman operator associated with a smooth map φ, the rate function can decay at most exponentially. To see this, note that for $k \in \mathbb{N}$ we have $\|Uf\|_k \leqslant c_k \|f\|_k$ for some finite constant $c_k = c_k(\|\varphi\|_{C^k}) > 1$. Iterating this n times, choosing $k = \lceil \beta \rceil$, and $g = U^n f$ in (1.2.4) gives

$$\|f\|^2 = \|U^n f\|^2 \leqslant h(n) \|f\|_\alpha \|f\|_k c_k^n \,,$$

forcing

$$h(n) \geqslant \frac{\|f\|^2 c_k^{-n}}{\|f\|_\alpha \|f\|_k} \,.$$

When the rate function h decays exponentially, the map φ is called exponentially mixing.

Definition 1.2.4. We say φ is α, β *exponentially mixing* if φ is strongly α, β mixing with an exponentially decaying rate function h.

We will shortly see that if φ is strongly exponentially mixing for some α, $\beta > 0$, then it must be strongly exponentially mixing for every α, $\beta > 0$.

Remark 1.2.5. By duality equation (1.2.4) implies that if φ is α, β mixing with rate function h, then

$$\|U^n f\|_{-\beta} \leqslant h(n) \|f\|_\alpha \,. \tag{1.2.5}$$

In particular, this implies $\|U^n f\|_{-\beta} \to 0$ as $n \to \infty$, and this has been used by many authors [MMP05, LTD11, Thi12, IKX14] to quantify (strong) mixing.

We now address the role of α, β in Definition 1.2.1. It turns out that if φ is strongly α, β mixing with rate function h, then it must be strongly α', β' mixing (at a particular rate) for *every* α', $\beta' > 0$. This is stated as the following proposition.

Proposition 1.2.6. *Suppose for some $\alpha, \beta > 0$, the map φ is strongly α, β mixing with rate function h. Then, for any α', $\beta' > 0$, the map φ is strongly α', β' mixing with rate function*

$$h'(t) \stackrel{\text{def}}{=} \lambda_1^{-\gamma} h(t)^{\delta} \,,$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on M,

$$\gamma \stackrel{\text{def}}{=} (\alpha' - \alpha)^+ + (\beta' - \beta)^+ + (\beta' \wedge \beta)\Big(1 - \frac{\alpha'}{\alpha}\Big)^+ + (\alpha' \wedge \alpha)\Big(1 - \frac{\beta'}{\beta}\Big)^+ \,,$$

$$\textit{and} \qquad \delta \stackrel{\text{def}}{=} \frac{(\alpha' \wedge \alpha)(\beta' \wedge \beta)}{\alpha\beta} \,.$$

In particular, if for some α, $\beta > 0$, φ is strongly α, β exponentially mixing, then it is strongly α', β' exponentially mixing for all α', $\beta' > 0$.

Proof. If $\beta \leqslant \beta'$, then we note

$$\begin{aligned} \|U^n f\|_{-\beta'} &\leqslant \lambda_1^{\beta - \beta'} \|U^n f\|_{-\beta} \\ &\leqslant \lambda_1^{\beta - \beta'} h(n) \|f\|_{\alpha} \,. \end{aligned}$$

On the other hand, if $\beta > \beta'$ then by Sobolev interpolation we have

$$\begin{aligned} \|U^n f\|_{-\beta'} &\leqslant \|U^n f\|_{-\beta}^{\beta'/\beta} \|U^n f\|^{1-\beta'/\beta} \\ &\leqslant h(n)^{\beta'/\beta} \|f\|_{\alpha}^{\beta'/\beta} \|f\|^{1-\beta'/\beta} \\ &\leqslant \lambda_1^{-\alpha(1-\beta'/\beta)} h(n)^{\beta'/\beta} \|f\|_{\alpha} \,. \end{aligned}$$

This shows that φ is strongly α, β' mixing with rate function

$$h_1(t) \stackrel{\text{def}}{=} \lambda_1^{-(\beta'-\beta)^+ - \alpha(1-\beta'/\beta)^+} h(t)^{(\beta'/\beta) \wedge 1} \,.$$

By dualizing, we see φ^{-1} is strongly β', α mixing with rate function h_1. Thus, using the above argument, φ^{-1} must be β', α' mixing with rate function

$$\begin{aligned} h'(t) &\stackrel{\text{def}}{=} \lambda_1^{-(\alpha'-\alpha)^+-\beta'(1-\alpha'/\alpha)^+} h_1(t)^{(\alpha'/\alpha)\wedge 1} \\ &= \lambda_1^{-\gamma} h(t)^{\delta}\,, \end{aligned}$$

as desired. □

We now turn our attention to weak mixing. Recall that the dynamical system generated by φ is said to be *weakly mixing* if for every pair of Borel sets $A, B \subseteq M$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Big| \operatorname{vol}(\varphi^{-k}(A) \cap B) - \operatorname{vol}(A) \operatorname{vol}(B) \Big| = 0\,. \tag{1.2.6}$$

Approximating by simple functions, and using the fact that U is L^2 bounded, one can show that (1.2.6) holds if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Big| \langle U^n f, g \rangle \Big|^2 = 0\,, \tag{1.2.7}$$

for all $f, g \in L^2_0$ (see for instance [EFHN15, Theorem 9.19 (iv)]). We can now quantify the *weak mixing rate* by by imposing a rate of convergence in (1.2.7).

Definition 1.2.7. Let $h\colon \mathbb{N} \to (0,\infty)$ be a decreasing function that vanishes at infinity. Given $\alpha, \beta \geqslant 0$, we say that φ is *weakly α, β mixing with rate function h* if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ and $n \in \mathbb{N}$ the associated Koopman operator U satisfies

$$\Big(\frac{1}{n} \sum_{k=0}^{n-1} \Big| \langle U^k f, g \rangle \Big|^2 \Big)^{1/2} \leqslant h(n) \|f\|_\alpha \|g\|_\beta\,. \tag{1.2.8}$$

Unlike Definition 1.2.1, the convergence rate need not involve stronger norms of *both* f and g. Indeed we will show later (Chapter 3.1) that for toral automorphisms, either α or β may be chosen to be 0. However, as we show next, it is impossible to choose *both* $\alpha = 0$ and $\beta = 0$, and thus convergence rate must involve a stronger norm of either f, or of g.

Proposition 1.2.8. *Let h be any function that decreases to* 0*. Then there does not exist any diffeomorphism φ which is weakly* 0, 0 *mixing with rate function h.*

Proof. Suppose for contradiction there exists a diffeomorphism φ which is weakly 0, 0 mixing with some rate function h. Recall, by definition, the rate function h must vanish at infinity. We will show that for any fixed $N \in \mathbb{N}$,

$$\sup_{\|f\|=\|g\|=1} \Big(\frac{1}{N} \sum_{k=0}^{n-1} |\langle U^k f, g\rangle|^2 \Big) = 1 \,. \tag{1.2.9}$$

This immediately implies $h(N) \geqslant 1$, contradicting the fact that h vanishes at ∞.

Thus to finish the proof we only need to prove (1.2.9). For this, note that φ must be weakly mixing (as h vanishes at infinity). Since weakly mixing maps are ergodic, we know (see for instance [Wal82]) that almost every point has a dense orbit. Let x_0 be one such point, and note that $\varphi^n(x_0) \neq x_0$ for all $n \neq 0$. By continuity of φ we can now find a $\delta = \delta(N) > 0$ such that

$$\varphi^n\Big(B(x_0,\delta)\Big) \cap \varphi^m\Big(B(x_0,\delta)\Big) = \emptyset \,, \quad \forall\ |n|, |m| < N \,,\ m \neq n \,.$$

Now let $\rho \in C_c(B(x_0,\delta) \cap L^2_0(M)$ be such that $\|\rho\| = 1$, and define the test functions f, g by

$$f = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} U^{-i} \rho \,, \qquad \text{and} \qquad g = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} U^{i} \rho \,.$$

Note by definition of ρ we have $\langle U^i \rho, U^j \rho\rangle = 0$ whenever $0 < |i-j| < N$. This implies $\|f\| = \|g\| = 1$, and

$$\frac{1}{N} \Big| \sum_{k=0}^{k} \langle U^k f, g\rangle \Big| = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{N} \sum_{i,j=0}^{N-1} \langle U^{k-i} \rho, U^j \rho\rangle = 1 \,.$$

This proves (1.2.9) as desired, finishing the proof. ☐

The analog of definition of mixing rates in continuous time is as follows.

Definition 1.2.9. Let $h\colon [0,\infty) \to (0,\infty)$ be a continuous, decreasing function that vanishes at ∞, and $\alpha, \beta \geqslant 0$. Let $\varphi_{s,t}\colon M \to M$ be the flow map of u defined by

$$\partial_t \varphi_{s,t} = u(\varphi_{s,t}, t) \qquad \text{and} \qquad \varphi_{s,s} = \mathrm{Id} \,.$$

1. We say that the vector field u is *strongly α, β mixing with rate function h* if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ we have

$$\left|\langle f \circ \varphi_{s,t}, g\rangle\right| \leqslant h(t-s)\|f\|_\alpha \|g\|_\beta \,. \tag{1.2.10}$$

2. We say that φ is *weakly α, β mixing with rate function h* if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ we have

$$\Big(\frac{1}{t-s}\int_s^t \left|\langle f \circ \varphi_{s,r}, g\rangle\right|^2 dr\Big)^{1/2} \leqslant h(t-s)\|f\|_\alpha \|g\|_\beta \,. \tag{1.2.11}$$

We devote the rest of this section to examples.

1.2.1 Example: Uniformly Expanding Map

Recall that the uniformly expanding map $\varphi\colon \mathbb{T}^1 \to \mathbb{T}^1$ is defined by

$$\varphi(x) = mx \mod 1\,.$$

for $m \in \mathbb{N}$ with $m \geqslant 2$.

Proposition 1.2.10. *The uniformly expanding map φ is 0, β exponentially mixing.*

Proof. Let U be the corresponding Koopman operator defined by φ. We find that for any $f \in L^2_0(\mathbb{T}^1)$

$$(Uf)^\wedge(k) = \begin{cases} \hat{f}\Big(\dfrac{k}{m}\Big) & \text{if } k = mp \text{ for some } p \in \mathbb{Z}\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence we have for any $\beta > 0$

$$\begin{aligned}\|U^n f\|^2_{H^{-\beta}} &= \sum_{k\neq 0} |m^n k|^{-\beta} |(U^n f)^\wedge(m^n k)|^2 = \sum_{k\neq 0} |m^n k|^{-\beta} |f)^\wedge(k)|^2 \\ &= m^{-\beta n}\|f\|^2_{H^{-\beta}} \leqslant m^{-\beta n}\|f\|^2_{L^2}\,.\end{aligned}$$

By duality, this shows that for any $f \in L^2_0(\mathbb{T}^1)$ and $g \in \dot{H}^\beta(\mathbb{T}^1)$, we have

$$|\langle U^n f, g\rangle| \leqslant 2^{-\beta n/2}\|f\|_{L^2}\|g\|_\beta\,.$$

This concludes the proof. ∎

Remark 1.2.11. Note, the above shows that the uniformly expanding map is strongly α, β mixing with $\alpha = 0$. The reason this does not contradict Remark 1.2.2 is because the expanding map is not a diffeomorphism, and thus the associated Koopman operator is not invertible. For any volume preserving diffeomorphism, we must of course have both $\alpha > 0$ and $\beta > 0$, as mentioned in Remark 1.2.2.

1.2.2 Example: Baker's map

Let $X = [0,1] \times [0,1]$ and define the map $\varphi : X \to X$

$$\varphi(x,y) = \begin{cases} (2x, \frac{1}{2}y) & \text{if } 0 \leqslant x < \frac{1}{2}\,, \\ (2x-1, \frac{1}{2}(y+1)) & \text{if } \frac{1}{2} \leqslant x \leqslant 1\,. \end{cases}$$

The Baker's map is strongly exponentially mixing and we refer the reader to [SOW06, Example 3.7.2] for a discussion, figures and a heuristic outline of a proof.

1.2.3 Example: Cat map

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The map φ, defined by

$$\varphi(x) = Ax \pmod{\mathbb{Z}^2}\,,$$

on the two dimensional torus is strongly mixing. Maps of this form are known as *toral automorphisms*, and have been extensively studied. We analyze them in Chapter 3, below, where we prove that they are exponentially mixing.

1.3 Dissipation Time

In our setup we will consider a mixing map on a closed Riemannian manifold. While the primary manifold we are interested in is the torus, there are, to the best of our knowledge, no known examples of smooth exponentially mixing maps on the torus that can be realized as the time one map of the flow of a smooth incompressible vector field. There are, however, several examples of closed Riemannian manifolds that admit such maps (see [Dol98, BW16] and references therein). Since working on closed Riemannian manifolds does not increase the complexity by much, we state our results in this context instead of restricting our attention to the torus.

Let M be a closed d-dimensional Riemannian manifold, and $\varphi\colon M \to M$ be a smooth volume preserving diffeomorphism. For simplicity we will subsequently assume that the volume form on M is normalized so that the total volume, $|M|$, is 1. Let $\nu > 0$ be the strength of the diffusion, Δ denote the Laplace-Beltrami operator on M, and $L^2_0 = L^2_0(M)$ denote the space of all mean zero square integrable functions on M. Given $\theta_0 \in L^2_0$, we consider the *pulsed diffusion* defined by

$$\theta_{n+1} = e^{\nu\Delta} U \theta_n \,. \tag{1.3.1}$$

Here $U\colon L^2(M) \to L^2(M)$ is the Koopman operator associated with φ, and is defined by $Uf = f \circ \varphi$. Our aim is to understand the asymptotic behaviour of the energy $\|\theta_n\|_{L^2_0}$ in the long time, small diffusivity limit. For notational convenience, we will use $\|\cdot\|$ to denote the L^2_0 norm, and $\langle \cdot, \cdot \rangle$ to denote the L^2_0 inner-product.

Since φ is volume preserving, the operator U is unitary and hence if $\nu = 0$ the system (1.3.1) conserves energy. If $\nu > 0$ and φ is mixing, then Koopman operator U produces fine scales which are rapidly damped by the diffusion. We quantify this using the notion of *dissipation time* in [FW03] (see also [FNW04, FNW06]).

Definition 1.3.1 (Dissipation time)**.** We define the *dissipation time* of the operator U by

$$\begin{aligned}
\tau_d &\stackrel{\text{def}}{=} \inf\Big\{ n \in \mathbb{N} \;\Big|\; \|(e^{\nu\Delta} U)^n\|_{L^2_0 \to L^2_0} < \frac{1}{e} \Big\} \\
&= \inf\Big\{ n \in \mathbb{N} \;\Big|\; \|\theta_n\| < \frac{\|\theta_0\|}{e} \quad \text{for all } \theta_0 \in L^2_0 \Big\} \,.
\end{aligned} \tag{1.3.2}$$

Since U is unitary we clearly have $\|\theta_n\| \leqslant e^{-\nu\lambda_1} \|\theta_{n-1}\|$, where $\lambda_1 > 0$ is the smallest non-zero eigenvalue of $-\Delta$ on M. Consequently, we always have

$$\tau_d \leqslant \frac{1}{\nu\lambda_1} \,, \tag{1.3.3}$$

Our aim is to investigate how (1.3.3) can be improved given an assumption on the "mixing rate" of φ.

In the continuous time setting, we consider the dissipation time for the advection-diffusion equation. Let M be a (smooth) closed Riemannian manifold, and u be a smooth, time dependent, divergence free vector field on M. Let θ be a solution to the advection-diffusion equation

$$\begin{cases} \partial_t \theta_s + (u(t) \cdot \nabla)\theta_s - \nu \Delta \theta_s = 0 & \text{in } M, \text{ for } t > s, \\ \theta_s(t) = \theta_{s,0} & \text{for } t = s. \end{cases} \tag{1.3.4}$$

for $t > s$, with initial data $\theta_s(s) = \theta_{s,0} \in L^2_0(M)$. Since u is divergence free we have

$$\frac{1}{2}\partial_t \|\theta_s(t)\|^2 + \nu \|\nabla \theta\|^2 = 0\,, \tag{1.3.5}$$

and hence

$$\|\theta_s(t)\| \leqslant e^{-\nu\lambda_1 (t-s)} \|\theta_{s,0}\|\,. \tag{1.3.6}$$

Our interest, again, is to to investigate how this decay rate can be quantifiably improved when the flow of u is mixing. At a first glance, this energy estimate (1.3.5) hides the information from the advection flow $u(x,t)$, which says the change of rate of $\|\theta\|_{L^2}$ only depends on $\|\nabla\theta\|_{L^2}$. We will show that the advection flow u does play an important role in the decay of $\|\theta\|_{L^2}$ by increasing the $\|\nabla\theta\|_{L^2}$ in some sense, which will be quantitatively studied later.

To give readers an intuitive idea of this, we refer to Figure 1.2 from paper [LTD11], which shows the snapshots of the scalar field evolution with initial distribution $\theta_0(x,y) = \sin x$ under a specifically constructed mixing flow on the domain $[0, 2\pi]^2$. We can see from Figure 1.2, under the influence of mixing, the space scales become smaller and smaller, which occurs along with energy moving to high frequencies. It thus then yields to a faster energy decay.

Similar to our treatment of pulsed diffusions, we define the *dissipation time* of u by

$$\begin{aligned} \tau_d &\stackrel{\text{def}}{=} \sup_{s \in \mathbb{R}} \Big(\inf \Big\{ t - s \;\Big|\; t \geqslant s, \text{ and } \|\theta_s(t)\| \leqslant \frac{\|\theta_{s,0}\|}{e} \quad \text{for all } \theta_{s,0} \in L^2_0 \Big\} \Big) \\ &= \sup_{s \in \mathbb{R}} \Big(\inf \Big\{ t - s \;\Big|\; t \geqslant s, \text{ and } \|\mathcal{S}_{s,t}\|_{L^2_0 \to L^2_0} \leqslant \frac{1}{e} \Big\} \Big)\,, \end{aligned} \tag{1.3.7}$$

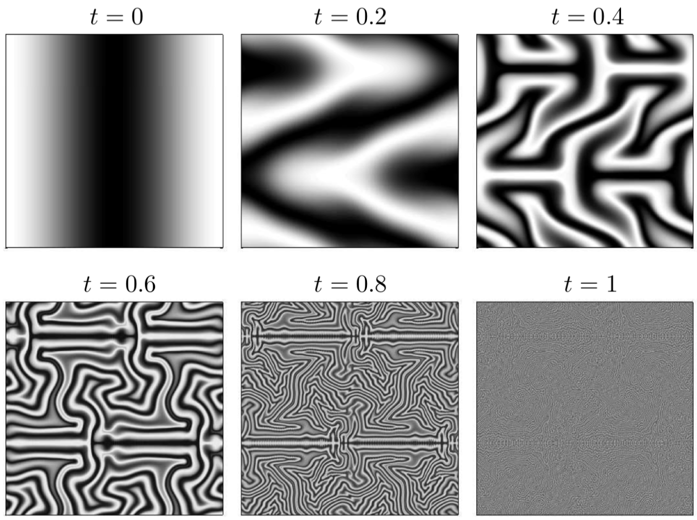

Figure 1.2: Evolution of scalar field in $[0, 2\pi]^2$ under an optimally mixing flow with initial $\theta_0(x, y) = \sin x$. This figure is taken from [LTD11].

where $\mathcal{S}_{s,t}$ is the solution operator to (1.3.4).

From (1.3.6) we immediately see that for any smooth divergence free advecting field u we again have

$$\tau_d \leqslant \frac{1}{\nu \lambda_1},$$

where λ_1 is the smallest non-zero eigenvalue of $-\Delta$ on M. If the flow of u is mixing, then we expect that τ_d to be much smaller than than $1/(\lambda_1 \nu)$. It turns out that all stationary vector fields for which $\nu \tau_d \to 0$ can be elegantly characterized in terms of the spectrum of the operator $u \cdot \nabla$. Indeed, seminal work of Constantin et. al. [CKRZ08] shows[2] that for time independent incompressible vector fields u, $\nu \tau_d \to 0$ if and only if the operator $(u \cdot \nabla)$ has no eigenfunctions in $\dot{H}^1$. Consequently, it follows that if the flow generated by u is weakly mixing, we must have $\nu \tau_d \to 0$ as $\nu \to 0$.

Our aim is to obtain bounds on the rate at which $\nu \tau_d \to 0$, under an assumption on the rate at which the flow of u mixes.

[2] More precisely, in [CKRZ08] the authors show that an incompressible, time independent, vector field u is *relaxation enhancing* if and only if $(u \cdot \nabla)$ has no eigenfunctions in $\dot{H}^1$. It is, however, easy to see that a vector field is relaxation enhancing if and only if $\nu \tau_d \to 0$.

1.4 Main results

The main results of this thesis quantifies the interaction between mixing and diffusion by studying the energy dissipation rate. Roughly speaking, our main results can be stated as follows:

1. In the continuous time setting we show (Theorem 4.1.1) that if the flow is strongly mixing, then the *dissipation time* (i.e. the time required for the system to dissipate a constant fraction of its initial energy) can be bounded explicitly in terms of the mixing rate. In particular, for exponentially mixing flows, then the dissipation time is bounded by $C|\ln \nu|^2$, where ν is the strength of the diffusion. If instead the flow is weakly mixing at a polynomial rate, then the dissipation time is bounded by C/ν^δ for some explicit $\delta \in (0,1)$ (Theorem 4.2.1).

2. Under similar assumptions in the discrete time setting we obtain the same bounds on the dissipation time (Theorems 2.1.1 and 2.2.1). We also show (Theorem 3.2.1) that the energy can not decay faster than double exponentially in time. Moreover, we obtain a family of examples where the energy indeed decays double exponentially in time. (In the continuous time setting the double exponential lower bound is known [Poo96], however, to the best of our knowledge there are no smooth flows which are known to attain this lower bound.)

3. In bounded domains, Berestycki et. al. [BHN05] studied asymptotics of the principal eigenvalue of the operator $-\nu\Delta + u \cdot \nabla$ as $\nu \to 0$. We show (Proposition 4.4.1) that one can use the dissipation time to obtain quantitative bounds on the rate at which the principal eigenvalue approaches 0.

Remark 1.4.1. In the continuous time setting similar results were obtained by Coti Zelati et al. [CZDE18], and their work is discussed in Section 4, below.

In this thesis, we will study the dissipation enhancement for *pulsed diffusions* in Chapter 2, and obtain an explicit formula bounding the dissipation time in terms of the mixing rate. In Chapter 3 we study dissipation enhancement and energy decay for pulsed diffusions under *toral automorphisms*, and obtain an family of examples where the energy decays double exponentially in time. In chapter 4, we study the dissipation enhancement for the advection diffusion equations. Finally, in Appendix A we characterize pulsed diffusions whose dissipation time vanishes faster than $O(1/\nu)$.

Chapter 2

Dissipation Enhancement for Pulsed Diffusions

In this chapter we study the *dissipation time* (1.3.2) for *pulsed diffusions* (1.3.1). We will give upper bounds of the dissipation time for both strongly and weakly mixing maps. Our result shows that given any mixing rate of a map, the upper bound of dissipation time for the corresponding pulsed diffusion model can be formulated explicitly. We will also briefly discuss the relations between the strongly and weakly mixing cases after that and give a characterization of the dissipation time on the map which has no $\dot{H}^1$ eigenfunctions.

2.1 The Strongly Mixing Case

First recall that the pulsed diffusion is defined by

$$\theta_{n+1} = e^{\nu\Delta} U \theta_n \,,$$

where $\theta_0 \in L^2_0$ and U is the Koopman operator associated with φ. Our main results on the dissipation time when the underlying map is strongly mixing are as follows:

Theorem 2.1.1. *Let $\alpha, \beta > 0$, and $h\colon [0,\infty) \to (0,\infty)$ be a decreasing function that vanishes at infinity. If φ is strongly α, β mixing with rate function h, then the dissipation time is bounded by*

$$\tau_d \leqslant \frac{C}{\nu H_1(\nu)} \,. \tag{2.1.1}$$

Here C is a universal constant which can be chosen to be 34, and $H_1\colon (0,\infty) \to (0,\infty)$ is defined by

$$H_1(\mu) \stackrel{\text{def}}{=} \sup\Big\{\lambda \;\Big|\; h\Big(\frac{1}{2\sqrt{\lambda\mu}}\Big) \leqslant \frac{\lambda^{-(\alpha+\beta)/2}}{2}\Big\}. \tag{2.1.2}$$

Before proceeding further, we compute the dissipation time τ_d in two useful cases.

Corollary 2.1.2. *Let $\alpha, \beta, h, \varphi$ be as in Theorem 2.1.1.*

1. *If the mixing rate function $h\colon (0,\infty) \to (0,\infty)$ is the power law*

$$h(t) = \frac{c}{t^p}, \tag{2.1.3}$$

for some $p > 0$, then the dissipation time is bounded by

$$\tau_d \leqslant \frac{C}{\nu^\delta} \qquad \text{where } \delta \stackrel{\text{def}}{=} \frac{\alpha+\beta}{\alpha+\beta+p}, \tag{2.1.4}$$

and $C = C(c,\alpha,\beta,p) > 0$ is a finite constant

2. *If the mixing rate function $h\colon [0,\infty) \to (0,\infty)$ is the exponential function*

$$h(t) = c_1 \exp(-c_2 t), \tag{2.1.5}$$

for some constants $c_1, c_2 > 0$, then the dissipation time is bounded by

$$\tau_d \leqslant C|\ln\nu|^2, \tag{2.1.6}$$

and $C = C(c_1,c_2,\alpha,\beta) > 0$ is a finite constant

Remark 2.1.3. In the proof of Corollary 2.1.2 (page 19) we will see that the bound (2.1.6) can be improved to a bound of the form

$$\tau_d \leqslant C_0\Big(|\ln\nu| - C_1 \ln\Big|\ln\nu - \ln|\ln\nu|\Big|\Big)^2$$

for explicit constants C_0, C_1 depending only on c_1, c_2, α, β and the constant C appearing in (2.1.1). However, since C is not optimal, this improvement is not significant.

To prove Theorem 2.1.1, we need two lemmas below.

Lemma 2.1.4. *Given $\theta \in L^2_0$, define $\mathcal{E}_\nu\theta$ by*

$$\mathcal{E}_\nu\theta \stackrel{\text{def}}{=} \frac{1}{\nu}\Big\|(1 - e^{2\nu\Delta})^{1/2} U\theta\Big\|^2 . \tag{2.1.7}$$

If for $\theta_0 \in L^2_0$ and $c_0 > 0$ we have

$$\mathcal{E}_\nu\theta_0 \geqslant c_0\|\theta_0\|^2 , \tag{2.1.8}$$

then

$$\|\theta_1\|^2 \leqslant e^{-\nu c_0}\|\theta_0\|^2 .$$

Lemma 2.1.5. *Let $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. Let λ_N be the largest eigenvalue satisfying $\lambda_N \leqslant H_1(\nu)$, where we recall that H_1 is defined in (2.1.2). If*

$$\mathcal{E}_\nu\theta_0 < \lambda_N\|\theta_0\|^2 , \tag{2.1.9}$$

then for

$$m_0 = 2\Big\lceil h^{-1}\Big(\frac{1}{2}\lambda_N^{-(\alpha+\beta)/2}\Big)\Big\rceil \tag{2.1.10}$$

we have

$$\|\theta_{m_0}\|^2 \leqslant \exp\Big(-\frac{\nu H_1(\nu) m_0}{16}\Big)\|\theta_0\|^2 . \tag{2.1.11}$$

Here h^{-1} is the inverse function of h.

Momentarily postponing the proofs of Lemmas 2.1.4 and 2.1.5 we prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Choosing $c_0 = \lambda_N$ and repeatedly applying Lemmas 2.1.4 and 2.1.5 we obtain an increasing sequence of times n_k such that

$$\|\theta_{n_k}\|^2 \leqslant \exp\Big(-\frac{\nu H_1(\nu) n_k}{16}\Big)\|\theta_0\|^2 , \qquad \text{and} \qquad n_{k+1} - n_k \leqslant m_0 .$$

This immediately implies

$$\tau_d \leqslant \frac{32}{\nu H_1(\nu)} + m_0 . \tag{2.1.12}$$

Note by choice of λ_N we have

$$h\Big(\frac{1}{2\sqrt{\nu\lambda_N}}\Big) \leqslant \frac{\lambda_N^{-(\alpha+\beta)/2}}{2}\,.$$

And since h is decreasing, it further implies

$$h^{-1}\Big(\frac{\lambda_N^{-(\alpha+\beta)/2}}{2}\Big) \leqslant \frac{1}{2\sqrt{\nu\lambda_N}}\,.$$

By the choice of m_0, we then have

$$m_0 \leqslant \frac{1}{\sqrt{\nu\lambda_N}} \leqslant \frac{1}{\nu\lambda_N}\,. \tag{2.1.13}$$

Recall by Weyl's lemma (see for instance [MP49]) we know

$$\lambda_j \approx \frac{4\pi\,\Gamma(\frac{d}{2}+1)^{2/d}}{\mathrm{vol}(M)^{2/d}} j^{2/d}\,, \tag{2.1.14}$$

asymptotically as $j \to \infty$. This implies $\lambda_{j+1} - \lambda_j = o(\lambda_j)$. Using this, and the fact that $H_1(\nu) \to \infty$ as $\nu \to 0$, we must have

$$\frac{1}{2} H_1(\nu) \leqslant \lambda_N \leqslant H_1(\nu)\,, \tag{2.1.15}$$

when ν is sufficiently small. Substituting this in (2.1.13) gives

$$m_0 \leqslant \frac{2}{\nu H(\nu)}\,,$$

and using this in (2.1.12) yields the desired result. □

To prove Corollary 2.1.2, we only need to compute the function H_1 explicitly for the specific rate functions of interest.

Proof of Corollary 2.1.2. When the mixing rate function h is the power law as defined in (2.1.3), we compute

$$H_1(\nu) = \Big(\frac{4^{p-1}}{c^2\nu^p}\Big)^{\frac{1}{\alpha+\beta+p}}\,.$$

Substituting this into (2.1.1) yields (2.1.4) as desired.

When the mixing rate function h is the exponential function as defined in (2.1.5), we can not compute H_1 exactly, as (2.1.2) only yields

$$H_1(\nu) = \frac{c_2^2}{4\nu}\Big(\ln 2 + \ln c_1 + \frac{\alpha+\beta}{2}\ln H_1(\nu)\Big)^{-2}. \tag{2.1.16}$$

Since $H_1(\nu) \to \infty$ as $\nu \to 0$, we know $H_1(\nu) \geqslant 1$ for sufficiently small ν.

$$H_1(\nu) \leqslant \frac{C}{\nu},$$

for some constant $C = C(c_1, c_2, \alpha, \beta)$. Using this in (2.1.16) yields

$$H_1(\nu) \geqslant \frac{C}{\nu|\ln\nu|^2}.$$

Substituting this in (2.1.1) yields (2.1.6) as desired. This argument can also be iterated to obtain improved bounds as stated in Remark 2.1.3. □

It remains to prove Lemmas 2.1.4 and 2.1.5.

Proof of Lemma 2.1.4. Note first that (1.3.1) and (2.1.7) imply the energy equality

$$\begin{aligned}\|\theta_1\|^2 &= \sum_{i=1}^{\infty} e^{-2\nu\lambda_i}|\langle U\theta_0, e_i\rangle|^2 = \sum_{i=1}^{\infty}|\langle U\theta_0, e_i\rangle|^2 - \nu\mathcal{E}_\nu\theta_0\\ &= \|\theta_0\|^2 - \nu\mathcal{E}_\nu\theta_0.\end{aligned} \tag{2.1.17}$$

Now using (2.1.8) immediately implies

$$\|\theta_1\|^2 \leqslant (1 - c_0\nu)\|\theta_0\|^2 \leqslant e^{-c_0\nu}\|\theta_0\|^2. \tag{2.1.18}$$

□

In order to prove Lemma 2.1.5, we first need to estimate the difference between the pulsed diffusion and the underlying dynamical system. We do this as follows.

Lemma 2.1.6. *Let ϕ_n, defined by*

$$\phi_n = U^n\theta_0,$$

be the evolution of θ_0 under the dynamical system generated by φ. Then for all $n \geqslant 0$ we have

$$\|\theta_n - \phi_n\| \leqslant \sum_{k=0}^{n-1}\sqrt{\nu\mathcal{E}_\nu\theta_k}. \tag{2.1.19}$$

Proof. Since $\phi_n = U\phi_{n-1}$, we have

$$\begin{aligned}
\|\theta_n - \phi_n\| &\leqslant \|(e^{\nu\Delta} - 1)U\theta_{n-1}\| + \|U(\theta_{n-1} - \phi_{n-1})\| \\
&= \Big(\sum_{i=1}^{\infty}(e^{-\nu\lambda_i} - 1)^2 |\langle U\theta_{n-1}, e_i\rangle|^2\Big)^{1/2} + \|\theta_{n-1} - \phi_{n-1}\| \\
&\leqslant \Big(\sum_{i=1}^{\infty}(1 - e^{-2\nu\lambda_i}) |\langle U\theta_{n-1}, e_i\rangle|^2\Big)^{1/2} + \|\theta_{n-1} - \phi_{n-1}\| \\
&\leqslant \sqrt{\nu\mathcal{E}_\nu\theta_{n-1}} + \|\theta_{n-1} - \phi_{n-1}\|\,,
\end{aligned}$$

and hence (2.1.19) follows by induction. $\square$

We now prove Lemma 2.1.5.

Proof of Lemma 2.1.5. By (2.1.17), we have

$$\|\theta_{m_0}\|^2 = \|\theta_1\|^2 - \nu \sum_{m=1}^{m_0-1} \mathcal{E}_\nu\theta_m\,. \tag{2.1.20}$$

Thus the decay of $\|\theta_{m_0}\|$ is governed by the growth of $\sum_{m=1}^{m_0-1}\mathcal{E}_\nu\theta_m$. In order to estimate $\mathcal{E}_\nu\theta_m$ we claim

$$2\|\theta_{m+1}\|_1^2 \leqslant \mathcal{E}_\nu\theta_m \leqslant 2\|U\theta_m\|_1^2\,, \quad \text{for all } m \in \mathbb{N}\,. \tag{2.1.21}$$

Indeed, by definition of $\mathcal{E}_\nu$ (equation (2.1.7)) we have

$$\nu\mathcal{E}_\nu\theta_m = \sum_{k=1}^{\infty}\Big(1 - e^{-2\nu\lambda_k}\Big)|(U\theta_m)^\wedge(k)|^2\,,$$

where $(U\theta_m)^\wedge(k) \stackrel{\text{def}}{=} \langle U\theta_m, e_k\rangle$ is the k-th Fourier coefficient of $U\theta_m$, and e_k is the eigenfunction of the Laplacian corresponding to the eigenvalue λ_k. Now (2.1.21) follows from the inequalities

$$2\nu\lambda_k e^{-2\nu\lambda_k} \leqslant 1 - e^{-2\nu\lambda_k} \leqslant 2\nu\lambda_k\,.$$

We next claim that for all sufficiently small ν we have

$$\|\theta_1\|_1^2 < \lambda_N\|\theta_1\|^2\,. \tag{2.1.22}$$

To see this, note that (2.1.9) and (2.1.21) imply

$$\|\theta_1\|_1^2 \leqslant \frac{1}{2}\mathcal{E}_\nu\theta_0 < \frac{\lambda_N}{2}\|\theta_0\|^2\,. \tag{2.1.23}$$

Moreover, our choice of λ_N (in equation (2.1.2)) guarantees $\lambda_N \leqslant 1/(2\nu)$ for all ν sufficiently small. Thus

$$\|\theta_1\|^2 = \|\theta_0\|^2 - \nu \mathcal{E}_\nu \theta_0 \geqslant (1 - \nu\lambda_N)\|\theta_0\|^2 \geqslant \frac{1}{2}\|\theta_0\|^2 \,,$$

and substituting this in equation (2.1.23) gives (2.1.22) as claimed.

We now claim that for N and m_0 as in the statement of Lemma 2.1.5 we have

$$\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \frac{\lambda_N m_0}{8}\|\theta_1\|^2 \,. \tag{2.1.24}$$

Note equation (2.1.24) immediately implies (2.1.11). Indeed, by (2.1.20), we have

$$\begin{aligned}\|\theta_{m_0}\|^2 &\leqslant \Bigl(1 - \frac{\nu\lambda_N m_0}{8}\Bigr)\|\theta_1\|^2 \leqslant \exp\Bigl(-\frac{\nu\lambda_N m_0}{8}\Bigr)\|\theta_0\|^2 \\ &\leqslant \exp\Bigl(-\frac{\nu H_1(\nu) m_0}{16}\Bigr)\|\theta_0\|^2 \,,\end{aligned}$$

where last inequality followed from (2.1.15).

Thus it only remains to prove equation (2.1.24). For this we let ϕ_m, defined by

$$\phi_m = U^{m-1}\theta_1 \,,$$

be the evolution of θ_1 under the dynamical system generated by φ. Let $P_N \colon L^2_0 \to L^2_0$ be the orthogonal projection onto $\operatorname{span}\{e_1, \dots, e_N\}$. Using (2.1.21) we have

$$\begin{aligned}\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m &\geqslant \sum_{m=m_0/2}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant 2 \sum_{m=m_0/2}^{m_0-1} \|\theta_{m+1}\|_1^2 \\ &\geqslant 2\lambda_N \sum_{m=m_0/2}^{m_0-1} \|(I - P_N)\theta_{m+1}\|^2 \\ &\geqslant \lambda_N \Bigl(\sum_{m=m_0/2}^{m_0-1} \|(I - P_N)\phi_{m+1}\|^2 \\ &\qquad - 2 \sum_{m=m_0/2}^{m_0-1} \|(I - P_N)(\theta_{m+1} - \phi_{m+1})\|^2 \Bigr)\end{aligned}$$

$$\geqslant \lambda_N\Big(\frac{m_0}{2}\|\phi_1\|^2 - \sum_{m=m_0/2}^{m_0-1}\|P_N\phi_{m+1}\|^2 - 2\sum_{m=m_0/2}^{m_0-1}\|\theta_{m+1}-\phi_{m+1}\|^2\Big)\,. \tag{2.1.25}$$

Now using Lemma 2.1.6 we estimate the last term on the right of (2.1.25) by

$$\sum_{m=m_0/2}^{m_0-1}\|\theta_{m+1}-\phi_{m+1}\|^2 \leqslant \sum_{m=m_0/2}^{m_0-1}\Big(\sum_{l=1}^{m}\sqrt{\nu\mathcal{E}_\nu\theta_l}\Big)^2 \leqslant \sum_{m=m_0/2}^{m_0-1} m\nu\sum_{l=1}^{m}\mathcal{E}_\nu\theta_l$$
$$\leqslant \frac{m_0^2\nu}{2}\sum_{l=1}^{m_0-1}\mathcal{E}_\nu\theta_l\,. \tag{2.1.26}$$

For the second term on the right of (2.1.25) we note that since U is strongly α,β mixing with rate function h, we have

$$\|U^m f\|_{-\beta} \leqslant h(m)\|f\|_\alpha\,,$$

for every $f \in \dot{H}^\alpha$ (see also (1.2.5)). This implies

$$\sum_{m=m_0/2}^{m_0-1}\|P_N\phi_{m+1}\|^2 \leqslant \sum_{m=m_0/2}^{m_0-1}\lambda_N^\beta\|\phi_{m+1}\|_{-\beta}^2 \leqslant \sum_{m=m_0/2}^{m_0-1}\lambda_N^\beta h(m)^2\|\phi_1\|_\alpha^2$$
$$\leqslant m_0 h\Big(\frac{m_0}{2}\Big)^2\lambda_N^\beta\|\phi_1\|_\alpha^2 \leqslant m_0 h\Big(\frac{m_0}{2}\Big)^2\lambda_N^\beta\|\theta_1\|^{2-2\alpha}\|\theta_1\|_1^{2\alpha}$$
$$\leqslant m_0 h\Big(\frac{m_0}{2}\Big)^2\lambda_N^{\alpha+\beta}\|\theta_1\|^2\,, \tag{2.1.27}$$

where the last inequality followed from (2.1.22).

Substituting (2.1.26) and (2.1.27) in (2.1.25) we obtain

$$\sum_{m=1}^{m_0-1}\mathcal{E}_\nu\theta_m \geqslant \frac{m_0\lambda_N}{1+\lambda_N\nu m_0^2}\Big(\frac{1}{2} - h\Big(\frac{m_0}{2}\Big)^2\lambda_N^{\alpha+\beta}\Big)\|\theta_1\|^2\,. \tag{2.1.28}$$

Clearly, by choice of m_0 in (2.1.10), we know

$$h\Big(\frac{m_0}{2}\Big)^2\lambda_N^{\alpha+\beta} \leqslant \frac{1}{4}\,. \tag{2.1.29}$$

Moreover, using the definition of H_1 (2.1.2) and the fact that $\lambda_N \leqslant H_1(\nu)$, we see

$$\lambda_N\nu m_0^2 \leqslant 1\,. \tag{2.1.30}$$

Now using (2.1.29) and (2.1.30) in (2.1.28) implies (2.1.24). This finishes the proof of Lemma 2.1.5. □

2.2 The Weakly Mixing Case.

When φ is weakly mixing, the bounds we obtain for the dissipation time are weaker than that in Theorem 2.1.1. We state these results next.

Theorem 2.2.1. *Let $\alpha, \beta \geqslant 0$, and $h\colon [0,\infty) \to (0,\infty)$ be a decreasing function that vanishes at infinity. If φ is weakly α, β mixing with rate function h, then the dissipation time is bounded by*

$$\tau_d \leqslant \frac{C}{\nu H_2(\nu)}\,. \tag{2.2.1}$$

Here C is a universal constant which can be chosen to be 34, and $H_2\colon (0,\infty) \to (0,\infty)$ is defined by

$$H_2(\mu) \stackrel{\text{def}}{=} \sup\Big\{\lambda \Bigm| h\Big(\frac{1}{\sqrt{2\mu\lambda}}\Big) \leqslant \frac{1}{2\sqrt{\tilde c}}\lambda^{-(2\alpha+2\beta+d)/4}\Big\}\,, \tag{2.2.2}$$

where $\tilde c = \tilde c(M) > 0$ is a finite constant that only depends on the manifold M.

Remark 2.2.2. We will see in the proof of Theorem 2.2.1 that the constant $\tilde c$ can be determined by the asymptotic growth of the eigenvalues of the Laplacian on M. Explicitly, let $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. Then for any $\varepsilon \in (0,1)$ we can choose

$$\tilde c = (1+\varepsilon)\lim_{j\to\infty}\frac{\lambda_j^{d/2}}{j} = \frac{(1+\varepsilon)(4\pi)^{d/2}\,\Gamma(\frac{d}{2}+1)}{\operatorname{vol}(M)}\,.$$

The existence, and precise value, of the limit above is given by Weyl's lemma (see for instance [MP49]).

We now compute τ_d explicitly when the weak mixing rate function h decays polynomially.

Corollary 2.2.3. *Let $\alpha, \beta, h, \varphi$ be as in Theorem 2.2.1. If the mixing rate function h is the power law (2.1.3) for some $p \in (0, 1/2]$*[1]*, then the dissipation time is bounded by*

$$\tau_d \leqslant C\nu^{-\delta}\,, \qquad \text{where} \quad \delta \stackrel{\text{def}}{=} \frac{d+2\alpha+2\beta}{d+2p+2\alpha+2\beta}\,, \tag{2.2.3}$$

and $C = C(\varphi, M, s, \alpha, \beta)$ is some finite constant.

[1] We require $p \in (0, 1/2]$, instead of $p > 0$, as the weak mixing rate can never be faster than $1/\sqrt{n}$. This can be seen immediately by choosing $f = g$ in (1.2.8).

We now turn our attention to Theorem 2.2.1. The proof is very similar to the proof of Theorem 2.1.1, the only difference is that the analog of Lemma 2.2.4 is not as explicit.

Lemma 2.2.4. *Let λ_N be the largest eigenvalue of $-\Delta$ such that $\lambda_N \leqslant H_2(\nu)$, and suppose*

$$\mathcal{E}_\nu \theta_0 < \lambda_N \|\theta_0\|^2 .$$

Then,

$$\|\theta_{m_0}\|^2 \leqslant \exp\Big(-\frac{\nu H_2(\nu) m_0}{16}\Big) \|\theta_0\|^2 .$$

where

$$m_0 = \Big\lceil h^{-1}\Big(\frac{1}{2\sqrt{\tilde{c}}} \lambda_N^{-(d+2\alpha+2\beta)/4}\Big)\Big\rceil + 1 , \tag{2.2.4}$$

and $\tilde{c}$ is the constant in Theorem 2.2.1 and Remark 2.2.2.

Given Lemma 2.2.4, the proof of Theorem 2.2.1 is essentially the same as the proof of Theorem 2.1.1.

Proof of Theorem 2.2.1. Choosing $c_0 = \lambda_N$ and repeatedly applying Lemmas 2.1.4 and 2.2.4 we obtain an increasing sequence of times n_k such that

$$\|\theta_{n_k}\|^2 \leqslant \exp\Big(-\frac{\nu H_2(\nu) n_k}{16}\Big) \|\theta_0\|^2 , \qquad \text{and} \qquad n_{k+1} - n_k \leqslant m_0 .$$

This immediately implies

$$\tau_d \leqslant \frac{32}{\nu H_2(\nu)} + m_0 . \tag{2.2.5}$$

By the choice of m_0 and λ_N, we notice that

$$m_0 \leqslant \frac{1}{\sqrt{\nu \lambda_N}} \leqslant \frac{1}{\nu \lambda_N} \leqslant \frac{2}{\nu H_2(\nu)} .$$

This proves (2.2.1). ∎

Before proving Lemma 2.2.4, we prove Corollary 2.2.3.

Proof of Corollary 2.2.3. The proof only involves computing H_2 explicitly when h is given by the power law (2.1.3). Using (2.2.2) we see

$$H_2(\nu) = \Bigl(2^{(p+2)/2} c\sqrt{\tilde{c}}\Bigr)^{-4\delta'} \nu^{-2p\delta'}, \qquad \text{where} \quad \delta' \stackrel{\text{def}}{=} \frac{1}{2\alpha + 2\beta + 2p + d}.$$

Substituting this into (2.2.1) yields (2.2.3) as desired. □

It remains to prove Lemma 2.2.4.

Proof of Lemma 2.2.4. We first claim that (2.1.24) still holds if λ_N, m_0 chosen as in the statement of Lemma 2.2.4. Once (2.1.24) is established, then the remainder of the proof is identical to that of Lemma 2.1.5.

To prove (2.1.24), we observe that the lower bound (2.1.25) (from the proof of Lemma 2.1.5) still holds in this case. For last term on the right of (2.1.25), we use the bound (2.1.26). The only difference here is to estimate the second term using the weak mixing assumption (1.2.8) instead. Observe

$$\frac{1}{m_0} \sum_{m=0}^{m_0-1} \|P_N \phi_{m+1}\|^2 = \sum_{l=1}^{N} \frac{1}{m_0} \sum_{m=0}^{m_0-1} |\langle e_l, U^m \theta_1 \rangle|^2 .$$

Since φ is weak α,β-mixing with rate function h, (1.2.8) yields

$$\begin{aligned} \frac{1}{m_0} \sum_{m=0}^{m_0-1} |\langle e_l, U^m \theta_1 \rangle|^2 &\leqslant h(m_0-1)^2 \|\theta_1\|_\alpha^2 \lambda_l^\beta \leqslant h(m_0-1)^2 \lambda_N^\beta \|\theta_1\|_\alpha^2 \\ &\leqslant h(m_0-1)^2 \lambda_N^\beta \|\theta_1\|^{2-2\alpha} \|\theta_1\|_1^{2\alpha} \leqslant h(m_0-1)^2 \lambda_N^{\beta+\alpha} \|\theta_1\|^2 . \end{aligned}$$

Together with (2.1.22) this gives

$$\begin{aligned} \frac{1}{m_0} \sum_{m=0}^{m_0-1} \|P_N \phi_{m+1}\|^2 &\leqslant h(m_0-1)^2 N \lambda_N^{\beta+\alpha} \|\theta_1\|^2 \\ &\leqslant \tilde{c} h(m_0-1)^2 \lambda_N^{(d+2\alpha+2\beta)/2} \|\theta_1\|^2 , \end{aligned}$$

where the last inequality follows from the fact that $\tilde{c}\lambda_N^{d/2}/2 \leqslant N \leqslant \tilde{c}\lambda_N^{d/2}$ when N is sufficiently large. This yields[2]

$$\sum_{m=m_0/2}^{m_0-1} \|P_N \phi_{m+1}\|^2 \leqslant \sum_{m=0}^{m_0-1} \|P_N \phi_{m+1}\|^2 \|\theta_1\|^2 \tag{2.2.6}$$

[2]Note that in the proof of Lemma 2.1.5, we used

$$\sum_{1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \sum_{m_0/2}^{m_0-1} \mathcal{E}_\nu \theta_m$$

$$\leqslant \tilde{c} m_0 h(m_0-1)^2 \lambda_N^{(d+2\alpha+2\beta)/2} \|\theta_1\|^2 \,.$$

Substituting this and (2.1.26) in (2.1.25) gives

$$\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \frac{m_0 \lambda_N}{1+m_0^2 \nu \lambda_N} \Big(\frac{1}{2} - \tilde{c}\, h(m_0-1)^2 \lambda_N^{(d+2\alpha+2\beta)/2} \Big) \|\theta_1\|^2 \,. \tag{2.2.7}$$

Now, the choice of m_0 in (2.2.4) forces

$$\tilde{c}\, h(m_0-1)^2 \lambda_N^{(d+2\alpha+2\beta)/2} \leqslant \frac{1}{4} \,. \tag{2.2.8}$$

Moreover, using (2.2.2) and the fact that $\lambda_N \leqslant H_2(\nu)$, we see

$$\lambda_N \nu m_0^2 \leqslant 4 h^{-1} \Big(\frac{1}{2\sqrt{\tilde{c}}} \lambda_N^{-(d+2\alpha+2\beta)/4} \Big)^2 \nu \lambda_N \leqslant 1 \,. \tag{2.2.9}$$

Substituting (2.2.8) and (2.2.9) in (2.2.7) implies (2.1.24), which finishes the proof. □

2.3 Relations between our bounds on the dissipation time in the strong and weak mixing cases

Note that as $\nu \to 0$, both $H_1(\nu) \to \infty$ and $H_2(\nu) \to \infty$. Thus the bounds obtained in both Theorems 2.1.1 and 2.2.1, guarantee $\nu \tau_d \to 0$ as $\nu \to 0$, and hence are stronger than the elementary bound (1.3.3).

Notice that if φ is strongly α, β mixing with rate function h, then it is also weakly α, β mixing with rate function h_w, where $h_w \colon [0,\infty) \to (0,\infty)$ is any continuous decreasing function such that

$$h_w(n) \stackrel{\text{def}}{=} \Big(\frac{1}{n} \sum_{k=0}^{n-1} h(k)^2 \Big)^{1/2} \qquad \text{for every} \quad n \in \mathbb{N} \,.$$

---

and focussed on bounding the tail of the sum in order to effectively use the decay of h. In (2.2.6), however, using only the tail of the sum does not improve our final result, and we can directly sum over the entire history. We only do it here because it allows us to directly use last part of the proof of Lemma 2.1.5.

In this case, however, one immediately sees that the bound provided by Theorem 2.2.1 is weaker than that provided by Theorem 2.1.1.

In particular, suppose φ is strongly α, β mixing with rate function h given by the power law (2.1.3) for some $p \in (0, 1/2]$. Then φ is also weakly α, β mixing with rate function given by

$$h_w(t) = \begin{cases} \dfrac{C_p}{t^p} & p < 1/2\,, \\ \Big(\dfrac{C_p \ln(1+t)}{t}\Big)^{1/2} & p = 1/2\,, \end{cases}$$

for some constant $C_p = C_p(c, p)$. In this case Corollary 2.2.3 applies when $p < 1/2$, and asserts that the dissipation time τ_d is bounded by (2.2.3). This, however, is weaker than (2.1.4).

Before proceeding further, we note that Fannjiang et. al. [FNW04] (see also [FW03, FNW06]) also obtain bounds on the dissipation time τ_d assuming the time decay of the correlations of the *diffusive operator* $e^{\nu\Delta}U$ for sufficiently small ν. Explicitly they assume sufficient decay of $\langle (e^{\nu\Delta}U)^n f, g\rangle$ as $n \to \infty$, and then show that the dissipation time τ_d is at most $C/|\ln \nu|$. In contrast, our results only assume decay of the correlations of the operator U (without diffusion) as in Definition 1.2.1.

In continuous time, Constantin et. al. [CKRZ08] (see also [KSZ08]) characterized flows for which the dissipation time is $o(1/\nu)$. Their result can directly be adapted to pulsed diffusions as follows.

Proposition 2.3.1. *The Koopman operator U has no eigenfunctions in $\dot{H}^1$ if and only if*

$$\lim_{\nu \to 0} \nu \tau_d = 0\,.$$

Since the proof is a direct adaptation of [CKRZ08, KSZ08], we relegate it to Appendix A.

Chapter 3

Toral Automorphisms and the Energy Decay of Pulsed Diffusions

In this chapter, we study the energy decay of pulsed diffusions with toral automorphisms. Using results from algebraic number theory we show that toral automorphisms are exponentially mixing, and that the L^2 energy of the associated pulsed diffusion decays double exponentially.

Recall a *toral automorphism* is a map of the form

$$\varphi(x) = Ax \pmod{\mathbb{Z}^d}, \tag{3.0.1}$$

where $A \in SL_d(\mathbb{Z})$ is an integer valued $d \times d$ matrix with determinant 1. Maps of this form are known as *"cat maps"*, and one particular example is when $d = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

The reason for the somewhat unusual name is that originally "CAT" was an abbreviation for Continuous Automorphism of the Torus. However, it has now become tradition to demonstrate the mixing effects of this map using the image of a cat [SOW06].

3.1 Mixing Rates of Toral Automorphisms

It is well known that no eigenvalue of A is a root of unity, if and only if φ is ergodic, if and only if φ is strongly mixing (see [Kat71], Page 160, problem 4.2.11 in [KH95]) Our interest is in understanding the mixing rates in the sense of Definition 1.2.1.

Proposition 3.1.1. *Let $A \in SL_d(\mathbb{Z})$ be such that:*

(C1) No eigenvalue of A is a root of unity,

(C2) and the characteristic polynomial of A is irreducible over $\mathbb{Q}$.

If $\alpha, \beta > 0$ then the toral automorphism $\varphi \colon \mathbb{T}^d \to \mathbb{T}^d$ defined by (3.0.1) is strongly α, β mixing with rate function

$$h(n) = C_{\alpha,\beta} \exp\Big(-\frac{n}{C_0}\Big(\alpha \wedge \frac{\beta}{d-1}\Big)\Big)\,, \tag{3.1.1}$$

for some finite non-zero constants $C_{\alpha,\beta} = C_{\alpha,\beta}(A, \alpha, \beta)$ and $C_0 = C_0(A)$.

Remark 3.1.2. Condition (C2) above is equivalent to assuming that A has no proper invariant subspaces in $\mathbb{Q}^d$.

For completeness, we also mention that if A satisfies Condition (C1) above, then A is also weakly α, β mixing if either $\alpha = 0$ or $\beta = 0$ (but not both).

Proposition 3.1.3. *Let $A \in SL_d(\mathbb{Z})$ satisfy the condition (C1) in Proposition 3.1.1.*

1. *If either $\alpha > 0$ and $\beta = 0$, or $\alpha = 0$ and $\beta > 0$, then there exists a finite constant $C_{\alpha,\beta} = C(\alpha, \beta)$ such that φ is weakly α, β mixing with rate function*

$$h(n) = \begin{cases} \dfrac{C_{\alpha,\beta}}{\sqrt{n}}\,, & \alpha \vee \beta > \dfrac{d}{2}\,, \\ C_{\alpha,\beta}\Big(\dfrac{\ln n}{n}\Big)^{1/2}, & \alpha \vee \beta = \dfrac{d}{2}\,, \\ \dfrac{C_{\alpha,\beta}}{n^{(\alpha\vee\beta)/d}}\,, & \alpha \vee \beta < \dfrac{d}{2}\,. \end{cases} \tag{3.1.2}$$

2. *If further A satisfies condition (C2) in Proposition 3.1.3, and both $\alpha > 0$ and $\beta > 0$, then there exists a finite constant $C_{\alpha,\beta} = C(A, \alpha, \beta)$ such that φ is weakly α, β mixing with rate function*

$$h(n) = \frac{C_{\alpha,\beta}}{\sqrt{n}}\,. \tag{3.1.3}$$

When $d = 2$, Proposition 3.1.1 is well known and can be proved elementarily. In higher dimensions, a version of Proposition 3.1.1 was proved by Lind [Lin82, Theorem 6] using a lemma of Katznelson [Kat71, Lemma 3] on Diophantine approximation. Proposition 3.1.1 can also be deduced from the results on the algebraic structure of toral automorphisms developed in [FW03]. These arguments, however, rely on three sophisticated results from number theory: the Schmidt subspace theorem [Sch80], Minkowski's theorem on linear forms [New72, Chapter VI] and van der Waerdern's theorem on arithmetic progressions [vdW27, Luk48]. We will avoid using these results, and instead instead prove Proposition 3.1.1 directly using the following two algebraic lemmas. These lemmas will be reused subsequently in the proof of sharpness of the double exponential bound (3.2.1) in Theorem 3.2.1.

Lemma 3.1.4. *Suppose $A \in SL_d(\mathbb{Z})$ satisfies the assumptions (C1) and (C2) in Proposition 3.1.1. There exists a basis $\{v_1, \dots, v_d\}$ of $\mathbb{C}^d$ such that the following hold:*

1. *Each v_i is an eigenvector of A.*
2. *If $k \in \mathbb{Z}^d - 0$, and $a_i = a_i(k) \in \mathbb{C}$ are such that*

$$k = \sum_1^d a_i(k) v_i = \sum_1^d a_i v_i\,,$$

then we must have

$$\prod_{i=1}^d |a_i(k)| \geqslant 1\,. \tag{3.1.4}$$

Lemma 3.1.5 (Kronecker [Kro57])**.** *Let p be a monic polynomial with integer coefficients that is irreducible over $\mathbb{Q}$. If all the roots of p are contained in the unit disk, they must be roots of unity.*

The proofs of Lemma 3.1.4 and 3.1.5 use elementary facts about algebraic number fields, and to avoid breaking continuity, we defer the proofs to Section 3.3. The reason these lemmas arise here is as follows. Lemma 3.1.5 will guarantee that guarantee $(A^T)^{-1}$ has at least one eigenvalue, ζ_1, strictly outside the unit disk. Lemma 3.1.4 now guarantees that all non-zero Fourier frequencies have a certain minimum component in the eigenspace of ζ_1. This will of course dominate the long time behaviour, leading to exponential mixing of φ and rapid energy dissipation of the associated pulsed diffusion.

Proof of Proposition 3.1.1. Let $B = (A^T)^{-1}$, and $f \in L^2_0$. Observe

$$(Uf)^\wedge(k) = \int_{\mathbb{T}^d} e^{-2\pi i k\cdot x} f(Ax)\,dx = \int_{\mathbb{T}^d} e^{-2\pi i (Bk)\cdot x} f(x)\,dx = \hat{f}(Bk)\,,$$

and hence

$$(U^n f)^\wedge(k) = \hat{f}(B^n k)\,, \tag{3.1.5}$$

for all $n \geqslant 0$. Now to prove that φ is exponentially mixing, let $f \in \dot{H}^\alpha$, and $g \in \dot{H}^\beta$. Using (3.1.5) we have

$$\langle U^n f, g\rangle = \sum_{k\in\mathbb{Z}^d-0} \hat{f}(B^n k)\overline{\hat{g}(k)} = \sum_{k\in\mathbb{Z}^d-0} \frac{1}{|B^n k|^\alpha |k|^\beta} |B^n k|^\alpha \hat{f}(B^n k) |k|^\beta \overline{\hat{g}(k)}$$

Consequently

$$|U^n f, g| \leqslant \Big(\sup_{k\in\mathbb{Z}^d-0} \frac{1}{|B^n k|^\alpha |k|^\beta}\Big) \|f\|_\alpha \|g\|_\beta \tag{3.1.6}$$

We now estimate the pre-factor on the right of (3.1.6) using Lemmas 3.1.4 and 3.1.5. First note that $B \in SL_d(\mathbb{Z})$ also satisfies the assumptions (C1) and (C2). Let $v_1, \dots, v_d$ be the basis given by Lemma 3.1.4, and $\zeta_1, \dots, \zeta_d$ be the corresponding eigenvalues. Since the characteristic polynomial of B satisfies the conditions of Lemma 3.1.5, we see that B has at least one eigenvalue outside the unit disk. Without loss of generality we suppose $|\zeta_1| > 1$.

By equivalence of norms on finite dimensional spaces, we know there exists $c_* > 0$ such that

$$\frac{1}{c_*}|k'| \leqslant \Big(\sum |a_i(k')|^2\Big)^{1/2} \leqslant c_* |k'|\,, \quad \text{for all } k' \in \mathbb{Z}^d\,. \tag{3.1.7}$$

Using Lemma 3.1.4, we note

$$|B^n k| = \left|\sum a_i \zeta_i^n v_i\right| \geqslant \frac{|a_1||\zeta_1|^n}{c_*} \geqslant \frac{|\zeta_1|^n}{c_*|a_2|\cdots|a_d|} \geqslant \frac{|\zeta_1|^n}{c_*^d |k|^{d-1}}\,.$$

Thus

$$\sup_{k\in\mathbb{Z}^d-0} \frac{1}{|B^n k|^\alpha |k|^\beta} \leqslant |\zeta_1|^{-n\alpha}\Big(\sup_{k\in\mathbb{Z}^d-0} \frac{c_*^{d\alpha}}{|k|^{\beta-(d-1)\alpha}}\Big)\,.$$

If $(d-1)\alpha \leqslant \beta$, (3.1.6) and the above shows that φ is strongly α, β mixing with rate function $h(n) = C|\zeta_1|^{-n\alpha}$. This proves (3.1.1) in the case $(d-1)\alpha \leqslant \beta$.

On the other hand, if $(d-1)\alpha > \beta$, we let $\alpha' = \beta/(d-1)$. By the previous argument we know φ is α', β mixing with rate function $h(n) = C|\zeta_1|^{-n\alpha'}$. Since $\alpha > \alpha'$, $\|f\|_{\alpha'} \leqslant \|f\|_\alpha$ and it immediately follows that φ is also α, β mixing with the same rate function. This proves (3.1.1) when $(d-1)\alpha > \beta$ completing the proof. □

Proof of Proposition 3.1.3. The second assertion follows immediately from Proposition 3.1.1. Indeed, when both $\alpha, \beta > 0$, Proposition 3.1.1 implies φ is strongly α, β mixing with rate function h given by (3.1.1). Since the rate function decays exponentially, it is square summable and equation (3.1.3) holds with $C_{\alpha,\beta} = (\sum_{i=1}^\infty h(i)^2)^{1/2}$.

To prove the first assertion, suppose first $\alpha = 0$ and $\beta > 0$. As before set $B = (A^T)^{-1}$, and let $f, g \in L^2_0$ and observe

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 &= \frac{1}{n}\sum_{i=0}^{n-1}\Big|\sum_{k\in\mathbb{Z}^d-0} \hat{f}(B^i k)\overline{\hat{g}(k)}\Big|^2 \\ &\leqslant \frac{\|g\|_\beta^2}{n}\sum_{i=0}^{n-1}\sum_{k\in\mathbb{Z}^d-0}\frac{|\hat{f}(B^i k)|^2}{|k|^{2\beta}}\,. \end{aligned} \tag{3.1.8}$$

We now split the analysis into cases. First suppose $\beta > d/2$. By Kronecker's theorem (Lemma 3.1.5) we see that the matrix B can not have finite order, and hence k, Bk, B^2k, …, $B^{n-1}k$ are all distinct. Thus (3.1.8) implies

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{\|g\|_\beta^2}{n}\sum_{k\in\mathbb{Z}^d-0}\sum_{i=0}^{n-1}\frac{|\hat{f}(B^i k)|^2}{|k|^{2\beta}} \leqslant \frac{\|g\|_\beta^2}{n}\sum_{k\in\mathbb{Z}^d-0}\frac{\|f\|^2}{|k|^{2\beta}}\,.$$

Since $\beta > d/2$, the sum on the right is finite, showing φ is 0, β mixing with rate function $C/n^{1/2}$ as desired.

Suppose now $\beta < d/2$. Let $m \in \mathbb{N}$ be a large integer that will be chosen shortly, and split the above sum as

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{\|g\|_\beta^2}{n}\Big(\sum_{0<|k|\leqslant m}\sum_{i=0}^{n-1}\frac{|\hat{f}(B^i k)|^2}{|k|^{2\beta}} + \sum_{i=0}^{n-1}\sum_{|k|>m}\frac{|\hat{f}(B^i k)|^2}{|k|^{2\beta}}\Big) \tag{3.1.9}$$

$$\leqslant \|f\|^2\|g\|_\beta^2\Big[\Big(\frac{1}{n}\sum_{0<|k|\leqslant m}\frac{1}{|k|^{2\beta}}\Big) + \frac{1}{m^{2\beta}}\Big] \tag{3.1.10}$$

$$\leqslant \|f\|^2\|g\|_\beta^2\Big(\frac{Cm^{d-2\beta}}{n} + \frac{1}{m^{2\beta}}\Big)\,, \tag{3.1.11}$$

for some (explicit) constant $C = C(d)$, independent of n. (Note, we again used the fact that k, B^k, ..., are all distinct when computing the first sum on the right of (3.1.9) to obtain (3.1.10).) We now choose $m = Cn^{1/d}$ in order to minimize the right hand side. This implies

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{C\|f\|^2\|g\|_\beta^2}{n^{2\beta/d}}$$

proving (3.1.2) when $\beta < d/2$.

Finally, when $\beta = d/2$ we repeat the same argument above to obtain (3.1.10). When summed (3.1.10) now yields

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \|f\|^2\|g\|_\beta^2\Big(\frac{C\ln m}{n} + \frac{1}{m^d}\Big)\,, \tag{3.1.12}$$

and choosing $m = n$ yields (3.1.2) as desired.

We have now proved (3.1.2) when $\alpha = 0$ and $\beta > 0$. For the case $\alpha > 0$ and $\beta = 0$, note that $\langle U^i f, g\rangle = \langle f, U^{-i} g\rangle$. Thus replacing the matrix A with A^{-1} reduces the case when $\alpha > 0, \beta = 0$ to the case when $\alpha = 0$, $\beta > 0$. This finishes the proof. □

3.2 Double Exponential Energy Decay

We now turn to studying the energy decay as $n \to \infty$. Clearly

$$\|\theta_n\| \leqslant \Big\|\Big((e^{\nu\Delta}U)^{\tau_d}\Big)^{\lfloor n/\tau_d\rfloor}\theta_0\Big\| \leqslant \Big\|(e^{\nu\Delta}U)^{\tau_d}\Big\|^{\lfloor n/\tau_d\rfloor}\|\theta_0\| \leqslant e^{-\lfloor n/\tau_d\rfloor}\|\theta_0\|\,,$$

and thus the energy $\|\theta_n\|$ decays at least exponentially with rate $1/\tau_d$ as $n \to \infty$. This bound, however, is not optimal. Indeed, if φ is the Arnold cat map, it is known [TC03] that the energy decays double exponentially. We show that this remains true for a large class of toral automorphisms. Moreover, Poon [Poo96] proved a matching lower bound for the continuous time advection diffusion equation. This is readily adapted to the discrete time setting.

Theorem 3.2.1 (Energy decay)**.** *For any $\theta_0 \in \dot{H}^1$, there exist finite constants $C = C(\varphi) > 0$ and $\gamma = \gamma(\varphi) > 1$ for which the double exponential lower bound*

$$\|\theta_n\|^2 \geqslant \|\theta_0\|^2 \exp\Big(-\frac{C\nu\|\theta_0\|_1^2}{\|\theta_0\|^2}\gamma^n\Big)\,, \tag{3.2.1}$$

holds. Moreover, there exists a smooth, volume preserving diffeomorphism on the torus for which the above bound is achieved. Explicitly, if φ is any toral automorphism which has no proper invariant rational subspaces, and has no eigenvalues that are roots of unity, then there exists finite constants C and $\gamma > 1$ such that

$$\|\theta_n\|^2 \leqslant \|\theta_0\|^2 \exp\Big(-\frac{\nu\gamma^n}{C}\Big)\,, \tag{3.2.2}$$

for all $\theta_0 \in L^2_0$.

In the rest of this section, we aim to prove Theorem 3.2.1. Our first result shows that if a toral automorphism satisfies conditions (C1) and (C2) in Proposition 3.1.1, then the energy of the associated pulsed diffusion decays double exponentially. This will prove sharpness of the lower bound (3.2.1) in Theorem 3.2.1. Following this we will prove lower bound (3.2.1) itself using a convexity argument.

Proposition 3.2.2. *Suppose $A \in SL_d(\mathbb{Z})$ satisfies the assumptions (C1) and (C2) in Proposition 3.1.1. Let φ be the associated toral automorphism defined in (3.0.1), and θ_n be the pulsed diffusion defined by (1.3.1). Then there exist constants $c > 0$ and $\gamma > 1$ such that*

$$\|\theta_n\| \leqslant \exp\Big(-\frac{\nu\gamma^n}{c}\Big) \tag{3.2.3}$$

Remark 3.2.3. In the proof of Proposition 3.2.2 we will see that the constant γ can be chosen to be

$$\gamma = \prod_{i=1}^{d} (|\zeta_i| \vee 1)^{2/d}$$

where $\zeta_1, \ldots, \zeta_d$ are the eigenvalues of A.

Proof. Using (3.1.5) we see

$$\hat{\theta}_{n+1}(k) = e^{-\nu|k|^2} \hat{\theta}_n(Bk) .$$

Setting $A_* = A^T$, iterating the above, squaring and summing in k gives

$$\|\theta_n\|^2 = \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-2\nu \sum_{j=1}^{n} |A_*^j k|^2 \Big) |\hat{\theta}_0(k)|^2 . \tag{3.2.4}$$

Observe that the matrix A_* also satisfies the conditions (C1) and (C2) in Proposition 3.1.1. Let $v_1, \ldots, v_d$ be the basis of $\mathbb{C}^d$ given by Lemma 3.1.4, and $\zeta_1, \ldots, \zeta_d$ be the corresponding eigenvalues. Now (3.2.4) implies

$$\begin{aligned}
\|\theta_n\|^2 &\leqslant \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*^2} \sum_{j=1}^{n} \sum_{i=1}^{d} |a_i|^2 |\zeta_i|^{2j} \Big) |\hat{\theta}_0(k)|^2 \\
&= \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*^2} \sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) \Big) |\hat{\theta}_0(k)|^2 \\
&\leqslant \|\theta_0\|^2 \sup_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*^2} \sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) \Big) . \qquad (3.2.5)
\end{aligned}$$

where c_* is the constant in (3.1.7).

We will now show that the last term decays double exponentially in n. Indeed, the inequality of the means implies

$$\begin{aligned}
\sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) &\geqslant d \Big(\prod_{i=1}^{d} |a_i|^2 \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) \Big)^{1/d} \\
&= d \Big(\prod_{i=1}^{d} |a_i|^2 \Big)^{1/d} \Big(\prod_{i=1}^{d} \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) \Big)^{1/d} \\
&\geqslant d \Big(\prod_{i=1}^{d} \Big(\frac{|\zeta_i|^{2(n+1)} - |\zeta_i|^2}{|\zeta_i|^2 - 1} \Big) \Big)^{1/d} , \qquad (3.2.6)
\end{aligned}$$

where the last inequality followed from Lemma 3.1.4. As in the proof of Proposition 3.1.1, Lemma 3.1.5 guarantees that $\max_i |\zeta_i| > 1$. The right hand side of (3.2.6) is of order $\prod_i (|\zeta_i| \vee 1)^{2n/d}$ and substituting this in (3.2.5) gives (3.2.3) as desired. □

We now prove Theorem 3.2.1.

Proof. Proposition 3.2.2 immediately shows that the double exponential upper bound equation (3.2.2) is achieved for the desired class of toral automorphisms. Thus it only remains to prove the double exponential lower bound (3.2.1). For this, observe

$$\begin{aligned}
\ln\|\theta_{n+1}\|^2 - \ln\|\theta_n\|^2 &= \ln\Big(\frac{\|\theta_{n+1}\|^2}{\|\theta_n\|^2}\Big) = \ln\Big(\frac{\|\theta_{n+1}\|^2}{\|U\theta_n\|^2}\Big) \\
&= \ln\Big(\frac{\sum_i e^{-2\nu\lambda_i} |\langle U\theta_n, e_i\rangle|^2}{\sum_i |\langle U\theta_n, e_i\rangle|^2}\Big)\,,
\end{aligned}$$

where we recall that λ_i are the eigenvalues of the Laplacian, and e_i's are the corresponding eigenfunctions. Using concavity of the logarithm and Jensen's inequality to bound the last term on the right we obtain

$$\begin{aligned}
\ln\|\theta_{n+1}\|^2 - \ln\|\theta_n\|^2 &\geqslant \frac{-2\nu \sum_i \lambda_i |\langle U\theta_n, e_i\rangle|^2}{\sum_i |\langle U\theta_n, e_i\rangle|^2} = -2\nu \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} \\
&\geqslant -2\nu \|\nabla\varphi\|_{L^\infty}^2 \frac{\|\theta_n\|_1^2}{\|\theta_n\|^2}\,. \qquad (3.2.7)
\end{aligned}$$

We now claim

$$\frac{\|\theta_n\|_1^2}{\|\theta_n\|^2} \leqslant \|\nabla\varphi\|_{L^\infty}^{2n} \frac{\|\theta_0\|_1^2}{\|\theta_0\|^2}\,. \qquad (3.2.8)$$

Note that substituting (3.2.8) in (3.2.7) and summing in n immediately implies (3.2.1). Thus to finish the proof we only need to prove (3.2.8).

For this we observe

$$\begin{aligned}
\frac{\|\theta_{n+1}\|_1^2}{\|\theta_{n+1}\|^2} - \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} &= \frac{\|\theta_{n+1}\|_1^2 \|U\theta_n\|^2 - \|\theta_{n+1}\|^2 \|U\theta_n\|_1^2}{\|\theta_n\|^2 \|U\theta_n\|^2} \\
&= \frac{1}{\|\theta_n\|^2 \|U\theta_n\|^2} \Big(\sum_{i,j} e^{-2\nu\lambda_i} (\lambda_i - \lambda_j) |\langle U\theta_n, e_i\rangle|^2 |\langle U\theta_n, e_j\rangle|^2 \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|\theta_n\|^2\|U\theta_n\|^2}\Big(\sum_{i<j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2 \\
&\qquad\qquad + \sum_{i>j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2\Big) \\
&\leqslant \frac{1}{\|\theta_n\|^2\|U\theta_n\|^2}\Big(\sum_{i<j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2 \\
&\qquad\qquad + \sum_{i>j} e^{-2\nu\lambda_j}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2\Big) \\
&= 0\,.
\end{aligned}
$$

Thus

$$
\frac{\|\theta_{n+1}\|_1^2}{\|\theta_{n+1}\|^2} \leqslant \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} = \frac{\|U\theta_n\|_1^2}{\|\theta_n\|^2} \leqslant \|\nabla\varphi\|_{L^\infty}^2 \frac{\|\theta_n\|_1^2}{\|\theta_n\|^2}\,,
$$

and iterating yields (3.2.8). This finishes the proof. □

3.3 Diophantine Approximation and Kronecker's Theorem

We now prove Lemmas 3.1.4 and 3.1.5. The proofs rely on standard facts on algebraic number fields, and we refer the reader to the books [Mar77] and [Rib01] for a comprehensive treatment.

Before beginning the proof of Lemma 3.1.4, we remark that a weaker version of it follows directly from the Schmidt subspace theorem [Sch80, Ch VI, Thm. 1B]. Explicitly, the Schmidt subspace theorem guarantees that for any $\varepsilon > 0$ we have

$$
\Big|\prod_{i=1}^d a_i(k)\Big| \geqslant \frac{1}{|k|^\varepsilon}\,,
$$

at all integer points $k \in \mathbb{Z}^d$, *except* on finitely many proper rational subspaces. To use the Schmidt subspace theorem in our context we would need to handle the exceptional subspaces. The approach taken by Fannjiang et. al. in [FW03] is to use van der Waerdern's theorem on arithmetic progressions [vdW27, Luk48] to construct an equivalent minimization problem whose minimizer is guaranteed to lie outside the exceptional subspaces. In our specific context we can directly prove the stronger bound (3.1.4), and avoid using the Schmidt subspace theorem entirely.

Proof of Lemma 3.1.4. Let p be the characteristic polynomial of A, and ζ_1, …, ζ_d be the roots of p. Let $F = \mathbb{Q}(\zeta_1, \dots, \zeta_d)$ and $\mathcal{G} = \mathrm{Gal}(F/\mathbb{Q})$ denote the Galois group. Let $G_i \subseteq \mathcal{G}$ be the group of field automorphisms that fix ζ_i, and $F_i = \{x \in F \mid \sigma(x) = x \ \forall \sigma \in G_i\}$ be the fixed field of G_i. Since $\det(A - \zeta_i I) = 0$, there must exist v_i in the F_i vector space F_i^d such that $Av_i = \zeta_i v_i$. Viewing v_i as an element of $\mathbb{C}^d$, we let $V \in GL_d(\mathbb{C})$ be the matrix with columns v_1, …, v_d. Dividing each v_i by a large integer if necessary, we may assume that each entry of V^{-1} is an algebraic integer. We claim that v_1, …, v_d is the desired basis.

To see this suppose $k = \sum a_i v_i$. By construction of the basis note that if $\sigma \in \mathcal{G}$ is such that $\sigma(\zeta_i) = \zeta_j$, then $\sigma(v_i) = v_j$. This implies that $\sigma(a_i) = a_j$. Note also that since the groups G_i are conjugate, they all have the same cardinality. Consequently

$$p_* \stackrel{\text{def}}{=} \prod_{\sigma \in \mathcal{G}} \sigma(a_1) = \Big(\prod_{i=1}^{d} a_i \Big)^m ,$$

where $m = |G_1|$. Thus p_* is in the fixed field of $\mathcal{G}$, and hence must be rational.

Further, since $a_i = (V^{-1}k) \cdot e_i$, each a_i must also be an algebraic integer. This forces p_* to be a rational algebraic integer, and hence an integer. By transitivity of the Galois group we see that if $a_i = 0$ for some i, then we must have $a_j = 0$ for all j. Thus p_* must be a non-zero integer if $k \neq 0$. Hence $|p_*| \geqslant 1$ and (3.1.4) follows. ◻

Lemma 3.1.5 is due to Kronecker [Kro57]. This result was improved by Stewart [Ste78] and Dobrowolski [Dob79]. More generally Lehmer's conjecture [Leh33] asserts that if ζ_1, …, ζ_d are the roots of p and the product $\prod(1 \vee |\zeta_i|)$ is smaller than an absolute constant μ (widely believed to be approximately $1.176\dots$), then each ζ_i is a root of unity. For our purposes, however, Kronecker's original result will suffice. Since the proof is short and elementary, we present it below.

Proof of Lemma 3.1.5. Let ζ_1, …, ζ_d be the roots of p. For any $n \in \mathbb{N}$, let p_n be the minimal monic polynomial satisfied by ζ_1^n. Since the Galois conjugates of ζ_1^n are precisely ζ_2^n, …, ζ_d^n, the coefficients of p_n are symmetric functions of ζ_1^n, …, ζ_d^n. By assumption $|\zeta_i| \leqslant 1$, which implies $|\zeta_i^n| \leqslant 1$, which in turn implies that the coefficients of p_n are uniformly bounded as functions of n. There are only finitely many polynomials with degree at most d, and uniformly bounded integer coefficients. Thus for some distinct $m, n \in \mathbb{N}$ we must have $p_m = p_n$. This forces $\zeta_1^m = \zeta_1^n$ showing ζ_1 is a root of unity. $\square$

Chapter 4

Dissipation Enhancement for the Advection Diffusion Equation

The main results of this chapter are Theorems 4.1.1 and 4.2.1, which bound the *dissipation time* (1.3.7) in the continuous time setting (1.3.4). These results are improvements of the results in the original paper [FI19]. The improvement was obtained by using a better estimate for the difference between the diffusive system and the underlying dynamical system taken from [CZDE18], and then following the proof in [FI19]. As a result the dissipation time bounds we obtain in the continuous setting match those previously obtained in the discrete time setting.

4.1 The Strongly Mixing Case

Theorem 4.1.1. *Let $\alpha, \beta > 0$, and $h\colon [0,\infty) \to (0,\infty)$ be a decreasing function that vanishes at infinity. If u is strongly α, β mixing with rate function h, then the dissipation time is bounded by*

$$\tau_d \leqslant \frac{C}{\nu H_3(\nu)}\,. \tag{4.1.1}$$

Here C is a universal constant which can be chosen to be 18, and $H_3\colon (0,\infty) \to (0,\infty)$ is defined by

$$H_3(\mu) = \sup\Big\{\lambda \;\Big|\; \sqrt{\lambda} h^{-1}\Big(\frac{\lambda^{-(\alpha+\beta)/2}}{2}\Big) \leqslant \frac{1}{64\sqrt{\mu\|\nabla u\|_{L^\infty}}}\Big\}, \tag{4.1.2}$$

where h^{-1} is the inverse function of h.

As before, we now compute H_3 explicitly for polynomial, and exponential rate functions. These special cases were previously obtained in [CZDE18].

Corollary 4.1.2. *Let α, β, u, h be as in Theorem 4.1.1.*

1. *If the mixing rate function h is the power law (2.1.3), then*

$$\tau_d \leqslant \frac{C}{\nu^\delta}, \qquad \text{where} \quad \delta \stackrel{\text{def}}{=} \frac{\alpha+\beta}{\alpha+\beta+p}, \tag{4.1.3}$$

   and $C = C(\alpha, \beta, c, \|\nabla u\|_{L^\infty})$ is a finite constant.

2. *If the mixing rate function h is the exponential (2.1.5), then*

$$\tau_d \leqslant C|\ln \nu|^2, \tag{4.1.4}$$

   and $C = C(\alpha, \beta, c_1, c_2, \|\nabla u\|_{L^\infty})$ is a finite constant.

As in Section 1.3, let $\theta_{s,0} \in L^2_0(M)$, let $\theta_s(t)$ be the solution of (1.3.4). By the energy inequality (1.3.5) we know

$$\|\theta_s(t)\|^2 = \|\theta_s(s)\|^2 \exp\Big(-2\nu \int_s^t \frac{\|\theta_s(r)\|_1^2}{\|\theta_s(r)\|^2}\, dr\Big).$$

Thus, $\|\theta_s(t)\|$ decays rapidly when the ratio $\|\theta_s(t)\|_1/\|\theta_s(t)\|$ remains large. Precisely, if for some $c_0 > 0$, we have

$$\|\theta_s(t)\|_1^2 \geqslant c_0\|\theta_s(t)\|^2, \qquad \text{for all } s \leqslant t \leqslant t_0,$$

then

$$\|\theta_s(t)\|^2 \leqslant e^{-2\nu c_0(t-s)}\|\theta_{s,0}\|^2, \qquad \text{for all } s \leqslant t \leqslant t_0. \tag{4.1.5}$$

As in the proof of Theorems 2.1.1 and 2.2.1, we will show that if the ratio $\|\theta_{s,0}\|_1/\|\theta_{s,0}\|$ is small, then the mixing properties of u will guarantee that for some later time $t_0 > s$, $\|\theta_s(t_0)\|$ becomes sufficiently small. This is the content of the following lemma.

Lemma 4.1.3. *Choose λ_N to be the largest eigenvalue satisfying $\lambda_N \leqslant H_3(\nu)$ where $H_3(\nu)$ is defined in (4.1.2). If*

$$\|\theta_{s,0}\|_1^2 < \lambda_N \|\theta_{s,0}\|^2 \,, \tag{4.1.6}$$

then we have

$$\|\theta_s(t_0)\|^2 \leqslant \exp\Big(-\frac{\nu H_3(\nu)(t_0 - s)}{8}\Big)\|\theta_{s,0}\|^2 \,. \tag{4.1.7}$$

at a time t_0 given by

$$t_0 \stackrel{\text{def}}{=} s + 2h^{-1}\Big(\frac{\lambda_N^{-(\alpha+\beta)/2}}{2}\Big) \,.$$

Momentarily postponing the proof of Lemma 4.1.3, we prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Choosing $c_0 = \lambda_N$ and repeatedly applying the inequality (4.1.5) and Lemma 4.1.3, we obtain an increasing sequence of times (t'_k), such that $(t'_k) \to \infty$ and

$$\|\theta_s(t'_k)\|^2 \leqslant \exp\Big(-\frac{\nu H_3(\nu)(t'_k - s)}{8}\Big)\|\theta_{s,0}\|^2 \,, \text{ and } t'_{k+1} - t'_k \leqslant t_0 \,.$$

This immediately implies

$$\tau_d \leqslant \frac{16}{\nu H_3(\nu)} + (t_0 - s) \,. \tag{4.1.8}$$

By choice of λ_N and t_0, we know that $t_0 - s \leqslant 1/(\nu\lambda_N) \leqslant 2/(\nu H_3(\nu))$ for ν sufficiently small. The last inequality followed from Weyl's lemma as in the proof Theorem 2.1.1 (equation (2.1.15)). This proves (4.1.1) as desired. □

We now compute H_3 explicitly when the mixing rate function decays exponentially, or polynomially.

Proof of Corollary 4.1.2. Suppose first the mixing rate function h satisfies the power law (2.1.3). In this case the inverse is given by $h^{-1}(t) = (c/t)^{1/p}$. Thus, by definition of H_3 (in (4.1.2)), we have

$$\sqrt{H_3(\nu)}\Big(2cH_3(\nu)^{(\alpha+\beta)/2}\Big)^{1/p} = \frac{1}{64\sqrt{\nu\|\nabla u\|_{L^\infty}}} \,.$$

Since $H_3(\nu) \to \infty$ as $\nu \to 0$, the above forces

$$H_3(\nu) \approx C\nu^{-p/(\alpha+\beta+p)} ,$$

asymptotically as $\nu \to 0$, for some constant $C = C(c, p, \alpha, \beta, \|\nabla u\|_{L^\infty})$. Using this in (4.1.1) yields (4.1.3) as desired.

Suppose now the rate function h is the exponential (2.1.5). Then we see $h^{-1}(t) = (\ln c_1 - \ln t)/c_2$. By the definition of H_3 in (4.1.2), we have

$$\sqrt{H_3(\nu)}\Big(\ln c_1 + \frac{\alpha+\beta}{2}\ln(H_3(\nu)) + \ln 2\Big) = \frac{c_2}{64\sqrt{\nu\|\nabla u\|_{L^\infty}}} ,$$

which implies

$$H_3(\nu) = O\Big(\frac{1}{\nu|\ln\nu|^2}\Big) ,$$

asymptotically as $\nu \to 0$. Substituting this in (4.1.1) yields (4.1.4) as desired. $\square$

It remains to prove Lemma 4.1.3. For this we will need a standard result estimating the difference between θ and solutions to the inviscid transport equation.

Lemma 4.1.4. *Let ϕ_s, defined by*

$$\phi_s = \theta_{s,0} \circ \varphi_{s,t} ,$$

be the evolution of $\theta_{s,0}$ under the dynamical system generated by $\varphi_{s,t}$. If $\theta_{s,0} \in \dot{H}^1(M)$, then for all $t \geqslant s$, we have

$$\begin{aligned} &\|\theta_s(t) - \phi_s(t)\|^2 \\ &\leqslant 2\sqrt{2\nu(t-s)}\|\theta_{s,0}\|\Big(2\|\nabla u\|_{L^\infty}\int_s^t \|\theta_s\|_{H^1}^2\,d\gamma + \|\theta_{s,0}\|_{H^1}^2\Big)^{1/2} . \end{aligned} \tag{4.1.9}$$

Proof. First multiplying (1.3.4) by $\Delta\theta_s(t)$ and integrate over time, we get

$$\frac{d}{dt}\|\theta_s\|_{H^1}^2 + 2\nu\|\theta_s\|_{H^2}^2 \leqslant 2\|\nabla u\|_{L^\infty}\|\theta_s\|_{H^1}^2 ,$$

which yields

$$2\nu\int_s^t \|\theta_s\|_{H^2}^2\,d\gamma \leqslant 2\|\nabla u\|_{L^\infty}\int_s^t \|\theta_s\|_{H^1}^2\,d\gamma + \|\theta_{s,0}\|_{H^1}^2 \tag{4.1.10}$$

Also we note that

$$\begin{aligned}\frac{d}{dt}\|\theta_s - \phi_s\|^2 &= 2\nu\langle \Delta\theta_s, \theta_s - \phi_s\rangle \\ &\leqslant 4\nu\|\theta_s\|_{H^2}\|\theta_{s,0}\|\,,\end{aligned}$$

where in the last inequality the energy conservation was used. Hence, for any $t \geqslant s$, we have

$$\begin{aligned}\|\theta_s(t) - \phi_s(t)\|^2 &\leqslant 4\nu\|\theta_{s,0}\| \int_s^t \|\theta_s\|_{H^2}\,d\gamma \\ &\leqslant 2\sqrt{2\nu(t-s)}\|\theta_{s,0}\|\Big(2\nu\int_s^t \|\theta_s(\gamma)\|_{H^2}^2\,d\gamma\Big)^{1/2}.\end{aligned}$$

Combining this with (4.1.10), we further get

$$\|\theta_s(t) - \phi_s(t)\|^2 \leqslant 2\sqrt{2\nu(t-s)}\|\theta_{s,0}\|\Big(2\|\nabla u\|_{L^\infty}\int_s^t \|\theta_s\|_{H^1}^2\,d\gamma + \|\theta_{s,0}\|_{H^1}^2\Big)^{1/2},$$

which ends the proof. $\square$

We can now prove Lemma 4.1.3.

Proof of Lemma 4.1.3. Integrating the energy equality (1.3.5) gives

$$\|\theta_s(t_0)\|^2 = \|\theta_{s,0}\|^2 - 2\nu\int_s^{t_0}\|\theta_s(r)\|_1^2\,dr\,. \tag{4.1.11}$$

We claim that our choice of λ_N and t_0 will guarantee

$$\int_s^{t_0}\|\theta_s(r)\|_1^2\,dr \geqslant \frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{8}\,. \tag{4.1.12}$$

This immediately yields (4.1.7) since when ν is small enough, we have

$$\frac{1}{2}H_3(\nu) \leqslant \lambda_N \leqslant H_3(\nu)\,.$$

Thus to finish the proof we only have to prove (4.1.12). We prove by assuming the converse inequality holds $\int_s^{t_0}\|\theta_s(r)\|_1^2\,dr < \frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{8}$.

Note first

$$\int_s^{t_0}\|\theta_s(r)\|_1^2\,dr \geqslant \lambda_N\int_{\frac{t_0+s}{2}}^{t_0}\|(I-P_N)\theta_s(r)\|^2\,dr$$

$$\begin{aligned}
&\geqslant \frac{\lambda_N}{2}\int_{\frac{t_0+s}{2}}^{t_0} \|(I-P_N)\phi_s(r)\|^2\,dr \\
&\qquad - \lambda_N \int_{\frac{t_0+s}{2}}^{t_0} \|(I-P_N)\big(\theta_s(r)-\phi_s(r)\big)\|^2\,dr \\
&\geqslant \frac{\lambda_N(t_0-s)}{4}\|\theta_{s,0}\|^2 - \frac{\lambda_N}{2}\int_{\frac{t_0+s}{2}}^{t_0}\|P_N\phi_s(r)\|^2\,dr \\
&\qquad - \lambda_N\int_s^{t_0}\|\theta_s(r)-\phi_s(r)\|^2\,dr\,.
\end{aligned} \tag{4.1.13}$$

We will now bound the last two terms in (4.1.13). For the second term, note the strong mixing assumption gives

$$\begin{aligned}
&\int_{\frac{t_0+s}{2}}^{t_0}\|P_N\phi_s(r)\|^2\,dr \leqslant \lambda_N^\beta \int_{\frac{t_0+s}{2}}^{t_0}\|\phi_s(r)\|_{-\beta}^2\,dr \leqslant \lambda_N^\beta\int_{\frac{t_0+s}{2}}^{t_0} h(r-s)^2\|\theta_{s,0}\|_\alpha^2\,dr \\
&\quad\leqslant \frac{t_0-s}{2}\lambda_N^\beta h\Big(\frac{t_0-s}{2}\Big)^2\|\theta_{s,0}\|_\alpha^2 \leqslant \frac{t_0-s}{2}\lambda_N^\beta h\Big(\frac{t_0-s}{2}\Big)^2\|\theta_{s,0}\|^{2-2\alpha}\|\theta_{s,0}\|_1^{2\alpha}\,.
\end{aligned} \tag{4.1.14}$$

Using the assumption (4.1.6), we obtain

$$\int_{\frac{t_0+s}{2}}^{t_0}\|P_N\phi_s(r)\|^2\,dr \leqslant \frac{t_0-s}{2}\lambda_N^{\alpha+\beta}h\Big(\frac{t_0-s}{2}\Big)^2\|\theta_{s,0}\|^2\,. \tag{4.1.15}$$

Now we bound the last term in (4.1.13). Using Lemma 4.1.4 we obtain

$$\begin{aligned}
&\int_s^{t_0}\|\theta_s(r)-\phi_s(r)\|^2\,dr \\
&\quad\leqslant \int_s^{t_0} 2\sqrt{2\nu(t-s)}\|\theta_{s,0}\|\Big(2\|\nabla u\|_{L^\infty}\int_s^r\|\theta_s(t)\|_{H^1}^2\,dt + \|\theta_{s,0}\|_{H^1}^2\Big)^{1/2}\,dr \\
&\quad\leqslant 2\sqrt{2\nu}(t_0-s)^{3/2}\|\theta_{s,0}\|\Big(2\|\nabla u\|_{L^\infty}\int_s^{t_0}\|\theta_s(t)\|_{H^1}^2\,dt + \|\theta_{s,0}\|_{H^1}^2\Big)^{1/2} \\
&\quad\leqslant 2\sqrt{2\nu}(t_0-s)^{3/2}\|\theta_{s,0}\|^2\Big(\frac{\|\nabla u\|_{L^\infty}\lambda_N(t_0-s)}{4}+\lambda_N\Big)^{1/2}\,.
\end{aligned} \tag{4.1.16}$$

Now going back to (4.1.13) we get

$$\frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{8} > \frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{4} - \frac{t_0-s}{4}\lambda_N^{\alpha+\beta+1}h\Big(\frac{t_0-s}{2}\Big)^2\|\theta_{s,0}\|^2$$

$$- 2\lambda_N\sqrt{2\nu}(t_0-s)^{3/2}\|\theta_{s,0}\|^2\Big(\frac{\|\nabla u\|_{L^\infty}\lambda_N(t_0-s)}{4}+\lambda_N\Big)^{1/2}.$$

Plugging in the choice of t_0 and canceling $\lambda_N(t_0-s)\|\theta_{s,0}\|^2$ both sides, we get

$$\begin{aligned}\frac{1}{8} &> \frac{1}{4}-\frac{1}{16}-2\sqrt{2\nu}(t_0-s)^{1/2}\Big(\frac{\|\nabla u\|_{L^\infty}\lambda_N(t_0-s)}{4}+\lambda_N\Big)^{1/2}\\ &> \frac{1}{4}-\frac{1}{16}-2\sqrt{2\nu}(t_0-s)^{1/2}\Big(\frac{\|\nabla u\|_{L^\infty}\lambda_N(t_0-s)}{2}\Big)^{1/2}\\ &= \frac{1}{4}-\frac{1}{16}-2\sqrt{\|\nabla u\|_{L^\infty}\nu\lambda_N}(t_0-s)\,.\end{aligned}$$

According to the choice of λ_N, the right hand side is greater than $\frac{1}{8}$, which ends at a contradiction. $\square$

4.2 The Weakly Mixing Case.

In this section, we bound the dissipation time for weakly mixing flows.

Theorem 4.2.1. *Let $\alpha, \beta > 0$, and $h\colon [0,\infty) \to (0,\infty)$ be a decreasing function that vanishes at infinity. If u is strongly α, β mixing with rate function h, then the dissipation time is bounded by*

$$\tau_d \leqslant \frac{C}{\nu H_4(\nu)}\,. \tag{4.2.1}$$

Here C is a universal constant which can be chosen to be 18, and $H_4\colon (0,\infty) \to (0,\infty)$ is defined by

$$H_4(\mu) = \sup\Big\{\lambda \;\Big|\; \sqrt{\lambda}h^{-1}\Big(\frac{1}{2\sqrt{\tilde{c}}}\lambda^{-(d+2\alpha+2\beta)/4}\Big) \leqslant \frac{1}{64\sqrt{\nu\|\nabla u\|_{L^\infty}}}\Big\}, \tag{4.2.2}$$

where h^{-1} is the inverse function of h and $\tilde{c} = \tilde{c}(M) > 0$ is the same constant as in Theorem 2.2.1 and Remark 2.2.2.

As before, we compute the above dissipation time bound explicitly when the mixing rate function decays polynomially.

Corollary 4.2.2. *Suppose u is weakly α, β mixing with rate function h, where $\alpha, \beta > 0$, and h is power law (2.1.3). Then the dissipation time is bounded by*

$$\tau_d \leqslant C\nu^{-\delta}\,, \qquad \text{where} \quad \delta = \frac{d+2\alpha+2\beta}{d+2p+2\alpha+2\beta}\,, \tag{4.2.3}$$

and $C = C(c, \tilde{c}, \alpha, \beta, \|\nabla u\|_{L^\infty})$ is some finite constant.

We now turn our attention to Theorem 4.2.1. The proof is similar to the proof of Theorem 4.1.1. The main difference is that the analog of Lemma 4.1.3 is weaker.

Lemma 4.2.3. *Let λ_N to be the largest eigenvalue of $-\Delta$ such that $\lambda_N \leqslant H_4(\nu)$, where we recall that the function H_4 is defined in (4.2.2). If*

$$\|\theta_{s,0}\|_1^2 < \lambda_N \|\theta_{s,0}\|^2\,, \tag{4.2.4}$$

then we have

$$\|\theta_s(t_0)\|^2 \leqslant \exp\Big(-\frac{\nu H_4(\nu)(t_0-s)}{8}\Big)\|\theta_{s,0}\|^2\,, \tag{4.2.5}$$

at a time t_0 given by

$$t_0 = s + 2h^{-1}\Big(\frac{1}{2\sqrt{\tilde{c}}}\lambda_N^{-(d+2\alpha+2\beta)/4}\Big)\,.$$

Proof of Theorem 4.2.1. Given Lemma 4.2.3, the proof of Theorem 4.2.1 is identical to that of Theorem 4.1.1. □

As before, the proof of Corollary 4.2.2 only involves computing H_4 explicitly when the mixing rate function decays polynomially.

Proof of Corollary 4.2.2. When the mixing rate function h is given by the power law (2.1.3), we compute $h^{-1}(t) = (c/t)^{1/p}$. By the definition of H_4 (equation (4.2.2)), we have

$$\sqrt{H_4(\nu)}\Big(2c\sqrt{\tilde{c}}H_4(\nu)^{(d+2\alpha+2\beta)/4}\Big)^{1/p} = \frac{1}{64\sqrt{\nu\|\nabla u\|_{L^\infty}}}\,,$$

which then yields

$$H_4(\nu) = O(\nu^{-\frac{2p}{d+2\alpha+2\beta+2p}})$$

asymptotically as $\nu \to 0$. Substituting this in (4.2.1) yields (4.2.3) as desired. □

Proof of Lemma 4.2.3. Following the proof of Lemma 4.1.3, we claim that the inequality (4.1.12) still holds in our case, provided λ_N and t_0 are chosen correctly. Here we also prove by contradiction. Indeed, note that (4.1.13) and (4.1.16) still hold, and the only difference here is that we need to bound the second term in (4.1.13) using the weak mixing assumption. Explicitly, (1.2.11) gives

$$\begin{aligned}
\int_{\frac{t_0+s}{2}}^{t_0} \|P_N\phi_s(r)\|^2\,dr &\leqslant \int_{\frac{t_0+s}{2}}^{t_0} \sum_{l=1}^{N} |\langle \phi_s(r), e_l\rangle|^2\,dr \\
&\leqslant \sum_{l=1}^{N} \frac{t_0-s}{2} h\Big(\frac{t_0-s}{2}\Big)^2 \|\phi_s(0)\|_\alpha^2 \lambda_l^\beta \\
&\leqslant \frac{N(t_0-s)}{2} h\Big(\frac{t_0-s}{2}\Big)^2 \lambda_N^\beta \|\phi_{s,0}\|_\alpha^2 \\
&\leqslant \frac{N(t_0-s)}{2} h\Big(\frac{t_0-s}{2}\Big)^2 \lambda_N^{\alpha+\beta} \|\theta_{s,0}\|^2 \\
&\leqslant \frac{\tilde{c}(t_0-s)}{2} h\Big(\frac{t_0-s}{2}\Big)^2 \lambda_N^{(d+2\alpha+2\beta)/2} \|\theta_{s,0}\|^2\,,
\end{aligned} \tag{4.2.6}$$

where last inequality followed from the fact that $\frac{\tilde{c}}{2}\lambda_N^{d/2} \leqslant N \leqslant \tilde{c}\lambda_N^{d/2}$ holds when N is sufficiently large. Substituting (4.1.16) and (4.2.6) into (4.1.13), we obtain

$$\begin{aligned}
\frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{8} &> \int_s^{t_0} \|\theta_s(r)\|_1^2\,dr \\
&\geqslant \frac{\lambda_N(t_0-s)\|\theta_{s,0}\|^2}{4}\Big(1 - \tilde{c}\lambda_N^{(d+2\alpha+2\beta)/2} h\Big(\frac{t_0-s}{2}\Big)^2 \\
&\quad - 8\sqrt{2\nu(t_0-s)}(\|\nabla u\|_{L^\infty}\lambda_N(t_0-s)/4 + \lambda_N)^{1/2}\Big)\,.
\end{aligned}$$

By our choice of t_0, we have

$$\frac{1}{8} > \frac{1}{4}\Big(1 - \frac{1}{4} - 8\sqrt{\nu\|\nabla u\|_{L^\infty}\lambda_N}(t_0-s)\Big)$$

By the choice of λ_N, we have $8\sqrt{\nu\|\nabla u\|_{L^\infty}\lambda_N}(t_0-s) \leqslant 1/4$, which ends at a contradiction. This finishes the proof. □

4.3 Optimality

In the particular case of shear flows a stronger estimate on the dissipation time can be obtained using Theorem 1.1 in [BCZ17]. Namely let $u = u(y)$ be a smooth shear flow on the 2-dimensional torus with non-degenerate critical points, and let L_0^2 denote the space of all functions whose horizontal average is 0. Now Theorem 1.1 in [BCZ17] guarantees that the dissipation time is bounded by

$$\tau_d \leqslant C\frac{|\ln \nu|^2}{\nu^{1/2}}\,, \tag{4.3.1}$$

for some constant $C > 0$.

To place this in the context of our results, we restrict our attention to L_0^2 functions on $\mathbb{T}^2$ whose horizontal averages are all 0. On this space, the method of stationary phase one can show that the flow generated by u is strongly 1, 1 mixing with rate function $h(t) = Ct^{-1/2}$ (see equation (1.8) in [BCZ17]). Consequently, by Corollary 4.1.2 guarantees that the dissipation time is bounded by

$$\tau_d \leqslant \frac{C}{\nu^{4/5}}\,.$$

This, however, is weaker than (4.3.1).

In general, we recall that Poon [Poo96] showed the double exponential *lower bound*

$$\|\theta_s(t)\| \geqslant \exp\Bigl(-C\nu\gamma^{t-s}\Bigr)\|\theta_{s,0}\|\,, \tag{4.3.2}$$

for some constants $C > 0$ and $\gamma > 1$. To the best of our knowledge, there are no incompressible smooth divergence free vector fields for which the lower bound (4.3.2) is attained. Moreover, on the torus, recent work of Miles and Doering [MD18] suggests that the Batchelor length scale may limit the long term effectiveness of mixing forcing only a single-exponential energy decay.

4.4 The Principal Eigenvalue with Dirichlet Boundary Conditions

Finally, we turn our attention to studying the principal eigenvalue of the operator $-\nu\Delta + u \cdot \nabla$ in a bounded domain Ω with Dirichlet boundary conditions. In this case, in addition to u being smooth and divergence free, we also assume u is time independent and tangential on the boundary (i.e. $u \cdot \hat{n} = 0$ on $\partial\Omega$, where $\hat{n}$ denotes the outward pointing unit normal). Let $\mu_0(\nu, u)$ denote the principal eigenvalue of $-\nu\Delta + u \cdot \nabla$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$.

By Rayleigh's principle we note

$$\mu_0(\nu, u) \geqslant \mu_0(\nu, 0) = \nu\mu_0(1, 0)$$

where $\mu_0(1, 0)$ is the principal eigenvalue of the Laplacian. Our interest is in understanding the behaviour of $\mu_0(\nu, u)/\nu$ as $\nu \to 0$. Berestycki et. al. [BHN05] showed that $\mu_0(\nu, u)/\nu \to \infty$ if and only if $u \cdot \nabla$ has no first integrals in H^1_0. That is, $\mu_0(\nu, u)/\nu \to \infty$ if and only if there does not exist $w \in H^1_0(\Omega)$ such that $u \cdot \nabla w = 0$.

In general it does not appear to be possible to obtain a rate at which $\mu_0(\nu, u)/\nu \to \infty$. If, however, the flow generated by u is sufficiently mixing then we obtain a rate at which $\mu_0(\nu, u)/\nu \to \infty$ in terms of the mixing rate of u. This is our next result.

Proposition 4.4.1. *If u is a smooth, time independent, incompressible vector field which is tangential on $\partial\Omega$, then*

$$\frac{\mu_0(\nu, u)}{\nu} \geqslant \frac{1}{\nu\tau_d}\,. \tag{4.4.1}$$

Proposition 4.4.1 follows immediately by solving the advection diffusion equation with the principal eigenfunction as the initial data.

We now prove Proposition 4.4.1 estimating the principal eigenvalue of $-\nu\Delta + (u \cdot \nabla)$ in a bounded domain with Dirichlet boundary conditions.

Proof of Proposition 4.4.1. For notational convenience we will write μ_0 to denote $\mu_0(\nu, u)$. Let $\phi_0 = \phi_0(\nu, u)$ be the principal eigenfunction of the operator $-\nu\Delta + (u \cdot \nabla)$. Then we know

$$\psi(x, t) \stackrel{\text{def}}{=} \phi_0(x) e^{-\mu_0 t}$$

satisfies the advection diffusion equation

$$\partial_t \psi + u \cdot \nabla\psi - \nu\Delta\psi = 0\,,$$

with initial data ϕ_0. Consequently $\|\psi(t)\| = e^{-\mu_0 t}\|\psi(0)\|$. This forces $\tau_d \geqslant 1/\mu_0$ proving (4.4.1) as claimed. $\square$

We note that the proof of Theorems 4.1.1, 4.2.1 only use the spectral decomposition of the Laplacian, and are unaffected by the presence of spatial boundaries. Thus Theorems 4.1.1 and 4.2.1 still apply in this context. Consequently, if u is known to be (strongly, or weakly) mixing at a particular rate, then $\mu_0(\nu, u)/\nu$ must diverge to infinity, and the growth rate can be obtained by using (4.4.1) and Theorems 4.1.1, 4.2.1, or Corollaries 4.1.2, 4.2.2 as appropriate.

For example, if $\alpha, \beta > 0$ and u is strongly α, β mixing with the exponentially decaying rate function (2.1.5), then

$$\frac{\mu_0(\nu, u)}{\nu} \geqslant \frac{1}{C\nu |\ln \nu|^2}\,.$$

We remark, however, that in view of (4.3.2) and (4.4.1), we expect that if u that generates an exponentially mixing flow, then one should have

$$\frac{\mu_0(\nu, u)}{\nu} \geqslant \frac{1}{C\nu |\ln \nu|}\,.$$

We are, however, presently unable to prove this stronger bound.

Appendix A

A Characterization of Relaxation Enhancing Maps on the Torus

In continuous time, we call a flow is relaxation enhancing if $\|\mathcal{S}(\frac{1}{\nu})\| \to 0$ as $\nu \to 0$, where $\mathcal{S}(t)$ is the solution operator of the advection diffusion equation. In other words, a flow is relaxation enhancing if the dissipation time is $o(1/\nu)$. Constantin et. al. [CKRZ08] (see also [KSZ08]) characterized flows for which the dissipation time is $o(1/\nu)$. For our pulsed diffusion model, similar result still holds, which we stated before as Proposition 2.3.1. It says that the Koopman operator U has no eigenfunctions in $\dot{H}^1$ if and only if

$$\lim_{\nu \to 0} \nu \tau_d = 0\,.$$

We devote this appendix to proving Proposition 2.3.1. The main idea behind the proof is the same as that used in [CKRZ08, KSZ08]. The backward implication is simpler, and we present the proof of it first.

Proof of the backward implication in Proposition 2.3.1. For the backward implication, we need to assume $\nu\tau_d \to 0$, and show that the associated Koopman operator U has no non-constant eigenfunctions in $\dot{H}^1$. Suppose, for sake of contradiction, that $f \in \dot{H}^1$ is an eigenfunction, normalized so that $\|f\| = 1$, and let λ be the corresponding eigenvalue. Choosing $\theta_0 = f$, and defining θ_n by (1.3.1) we observe

$$|\langle \theta_{n+1}, f\rangle - \langle U\theta_n, f\rangle| = \left|\sum_k (1 - e^{-\nu\lambda_k})(U\theta_n)^\wedge(k)\overline{\hat{f}(k)}\right|$$

$$\leqslant \nu\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|(U\theta_n)^\wedge(k)|^2\Big)^{1/2}\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|\hat{f}(k)|^2\Big)^{1/2}$$
$$\leqslant \nu\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|(U\theta_n)^\wedge(k)|^2\Big)^{1/2}\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|\hat{f}(k)|^2\Big)^{1/2}$$
$$\leqslant \nu(\mathcal{E}_\nu\theta_n)^{1/2}\|f\|_1 \leqslant \frac{\nu}{2}\mathcal{E}_\nu\theta_n + \frac{\nu}{2}\|f\|_1^2\,.$$

Using equation (2.1.17), this gives

$$|\langle\theta_{n+1}, f\rangle - \langle U\theta_n, f\rangle| \leqslant \frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) + \frac{\nu}{2}\|f\|_1^2\,,$$

which implies

$$|\langle\theta_{n+1}, f\rangle| - |\langle U\theta_n, f\rangle| \geqslant -\frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) - \frac{\nu}{2}\|f\|_1^2\,.$$

Since $\langle U\theta_n, f\rangle = \langle\theta_n, U^* f\rangle = \lambda\langle\theta_n, f\rangle$, and $|\lambda| = 1$, the above implies

$$|\langle\theta_{n+1}, f\rangle| - |\langle\theta_n, f\rangle| \geqslant -\frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) - \frac{\nu}{2}\|f\|_1^2\,.$$

Iterating this gives

$$|\langle\theta_n, f\rangle| - |\langle f, f\rangle| \geqslant -\frac{1}{2}(\|f\|^2 - \|\theta_n\|^2) - \frac{n\nu}{2}\|f\|_1^2\,,$$

since $\theta_0 = f$. Thus

$$|\langle\theta_n, f\rangle| \geqslant \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\theta_n\|^2 - \frac{n\nu}{2}\|f\|_1^2 \geqslant \frac{1}{2} - \frac{n\nu}{2}\|f\|_1^2\,.$$

Now choosing n to be the dissipation time τ_d gives

$$\frac{1}{e} \geqslant |\langle\theta_{\tau_d}, f\rangle| \geqslant \frac{1}{2} - \frac{\tau_d\nu}{2}\|f\|_1^2\,,$$

and hence

$$\nu\tau_d \geqslant \frac{e-2}{e\|f\|_1^2}\,.$$

This contradicts the assumption $\nu\tau_d \to 0$ as $\nu \to 0$, finishing the proof. □

For the other direction, we need two lemmas. The first is an application of the discrete RAGE theorem.

Lemma A.0.1. *Let $K \subset S = \{\phi \in L_0^2 \mid \|\phi\| = 1\}$ be a compact set. Let P_c be the spectral projection on the continuous spectral subspace in the spectral decomposition of U. For any $N, \delta > 0$, there exists $n_c(N, \delta, K)$ such that for all $n \geqslant n_c$ and any $\phi \in K$, we have*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \|P_N U^i P_c \phi\|^2 \leqslant \delta \,. \tag{A.0.1}$$

Proof. Define

$$f(n, \phi) \stackrel{\text{def}}{=} \frac{1}{n-1} \sum_{i=1}^{n-1} \|P_N U^i P_c \phi\|^2 \,.$$

Recall that by the RAGE theorem [CFKS87] we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \|A U^i P_c \phi\|^2 = 0 \,, \quad \text{for any compact operator } A \,,$$

and hence for all ϕ, $f(\phi, n) \to 0$ as $n \to \infty$. Thus, to finish the proof, we only need to show that this convergence is uniform on compact sets.

To prove this, it is enough to prove the functions $f(\cdot, n)$ are equicontinuous. For this observe that for any $\phi_1, \phi_2 \in S$ we have

$$\begin{aligned}
&|f(n, \phi_1) - f(n, \phi_2)| \\
&\quad \leqslant \frac{1}{n-1} \sum_{i=1}^{n-1} \Big| \|P_N U^i P_c \phi_1\| - \|P_N U^i P_c \phi_2\| \Big| \Big(\|P_N U^i P_c \phi_1\| + \|P_N U^i P_c \phi_2\| \Big) \\
&\quad \leqslant \frac{1}{n-1} \sum_{i=1}^{n-1} \|\phi_1 - \phi_2\| \Big(\|\phi_1\| + \|\phi_2\| \Big) \\
&\quad \leqslant 2\|\phi_1 - \phi_2\| \,.
\end{aligned}$$

This shows equicontinuity, finishing the proof. □

Lemma A.0.2. *Assume that the Koopman operator U has no eigenfunctions in $\dot{H}^1$. Let P_p be the spectral projection on its point spectral subspace. Let K be a compact subset of S. Define the set $K_1 = \{\phi \in K \mid \|P_p \phi\| \geqslant \frac{1}{2}\}$. Then for any $C > 0$, there exist $N_p(C, K)$ and $n_p(C, K)$ such that for any $N \geqslant N_p(C, K)$, any $n \geqslant n_p(C, K)$, and any $\phi \in K_1$,*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \|P_N U^i P_p \phi\|_1^2 \geqslant C \,. \tag{A.0.2}$$

The proof of this is the same as Lemma 3.3 in [CKRZ08] and we do not present it here. We can now finish the proof of Proposition 2.3.1.

Proof of the forward implication in Proposition 2.3.1. For this direction we are given that U has no eigenfunctions in $\dot{H}^1$, and need to show $\nu\tau_d \to 0$ as $\nu \to 0$. We will show that for any $\eta > 0$,

$$\Big\|\theta\Big(\Big\lceil\frac{\eta}{\nu}\Big\rceil\Big)\Big\| \to 0 \text{ as } \nu \to 0\,, \tag{A.0.3}$$

which immediately implies $\nu\tau_d \to 0$ as $\nu \to 0$. To prove (A.0.3), we need to show for any given η, ε, there exists ν_0, such that for any $\nu \leqslant \nu_0$, we have $\|\theta(\lceil\frac{\eta}{\nu}\rceil)\|^2 \leqslant \varepsilon$ for any initial $\theta_0 \in H$ with $\|\theta_0\| = 1$. We choose N large enough satisfying $e^{-\lambda_N\eta/80} \leqslant \varepsilon$. Denote $K = \{\phi \in S \,|\, \|\phi\|^2 \leqslant \lambda_N\}$, and $K_1 = \{\phi \in K \,|\, \|P_p\phi\| \geqslant \frac{1}{2}\}$. Let n_1 be

$$n_1 = \max\Big\{2, n_p(5\lambda_N, K), n_c\Big(N, \frac{1}{20}, K\Big)\Big\}\,,$$

and choose ν_0 small enough so that

$$n_1 \leqslant \frac{\eta}{2\nu_0}\,, \qquad \nu_0 n_1^2 \leqslant \frac{1}{\lambda_N} \qquad \text{and} \qquad \frac{n_1^2\nu\lambda_N\|\nabla\varphi\|_{L^\infty}^{2n_1+2}}{(n_1-1)(\|\nabla\varphi\|_{L^\infty}^2-1)} \leqslant \frac{1}{4}\,.$$

Note that if $\mathcal{E}_\nu\theta_n \geqslant \lambda_N\|\theta_n\|^2$ for all $n \in [0, \lceil\eta/\nu\rceil]$, then we have

$$\Big\|\theta\Big(\Big\lceil\frac{\eta}{\nu}\Big\rceil\Big)\Big\|^2 \leqslant e^{-\nu\lambda_N\lceil\eta/\nu\rceil} \leqslant e^{-\lambda_N\eta} \leqslant \varepsilon\,.$$

If not, let $n_0 \in [0, \lceil\eta/\nu\rceil]$ be the first time satisfying $\mathcal{E}_\nu\theta_{n_0} < \lambda_N\|\theta_{n_0}\|^2$. Similar to (2.1.22) we have $\|\theta_{n_0+1}\|_1^2 < \lambda_N\|\theta_{n_0+1}\|^2$. We claim that our choice of n_1 will guarantee

$$\|\theta_{n_0+n_1}\|^2 \leqslant e^{-\lambda_N\nu n_1/40}\|\theta_{n_0}\|^2\,. \tag{A.0.4}$$

Given (A.0.4), we can find $\tilde{n} \in [\eta/(2\nu), \eta/\nu]$ such that $\|\theta(\lceil\eta/\nu\rceil)\|^2 \leqslant \|\theta_{\tilde{n}}\|^2 \leqslant e^{-\lambda_N\nu\tilde{n}/40} \leqslant e^{-\lambda_N\eta/80} \leqslant \varepsilon$, proving (A.0.3) as desired.

Thus it only remains to prove (A.0.4). For this, define $\phi_m = U^{m-1}\theta_{n_0+1}$, and observe

$$\frac{\phi_1}{\|\phi_1\|} = \frac{\theta_{n_0+1}}{\|\theta_{n_0+1}\|} \in K\,, \quad P_c\phi_m = U^{m-1}P_c\theta_{n_0+1}\,, \quad \text{and} \quad P_p\phi_m = U^{m-1}P_p\theta_{n_0+1}\,.$$

We now consider two cases.

Case I: $\|P_c\theta_{n_0+1}\|^2 \geqslant \frac{3}{4}\|\theta_{n_0+1}\|^2$ *(or equivalently* $\|P_p\theta_{n_0+1}\|^2 \leqslant \frac{1}{4}\|\theta_{n_0+1}\|^2$*).* In this case, we have

$$
\begin{aligned}
\sum_{m=1}^{n_1-1} \mathcal{E}_\nu \theta_{n_0+m} &\geqslant 2\sum_{m=1}^{n_1-1} \|\theta_{n_0+1+m}\|_1^2 \\
&\geqslant 2\lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)\theta_{n_0+1+m}\|^2 \\
&\geqslant \lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)\phi_{m+1}\|^2 - 2\lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)(\theta_{n_0+1+m}-\phi_{m+1})\|\,.
\end{aligned}
\tag{A.0.5}
$$

By direct calculation, we also have

$$
\begin{aligned}
\|(I-P_N)\phi_{m+1}\|^2 &\geqslant \frac{1}{2}\|(I-P_N)P_c\phi_{m+1}\|^2 - \|(I-P_N)P_p\phi_{m+1}\|^2 \\
&\geqslant \frac{1}{2}\|U^m P_c\theta_{n_0+1}\|^2 - \frac{1}{2}\|P_N U^m P_c\theta_{n_0+1}\|^2 - \|U^m P_p\theta_{n_0+1}\|^2 \\
&= \frac{1}{2}\|P_c\theta_{n_0+1}\|^2 - \frac{1}{2}\|P_N U^m P_c\theta_{n_0+1}\|^2 - \|P_p\theta_{n_0+1}\|^2\,.
\end{aligned}
$$

By Lemmas A.0.1,A.0.2, and the choice of n_1, we have

$$
\frac{1}{n_1-1}\sum_{m=1}^{n_1-1} \|(I-P_N)\phi_{m+1}\|^2 \geqslant \frac{1}{10}\|\theta_{n_0+1}\|^2\,. \tag{A.0.6}
$$

Substituting (2.1.26) and (A.0.6) in (A.0.5) gives

$$
\sum_{m=1}^{n_1-1} \mathcal{E}_\nu \theta_{n_0+m} \geqslant \frac{\lambda_N(n_1-1)}{20}\|\theta_{n_0+1}\|^2\,.
$$

Since $\|\theta_{n_0+n_1}\|^2 = \|\theta_{n_0+1}\|^2 - \nu\sum_{m=1}^{n_1-1}\mathcal{E}_\nu\theta_{n_0+m}$, we further have

$$
\begin{aligned}
\|\theta_{n_0+n_1}\|^2 &\leqslant \Big(1 - \frac{\nu\lambda_N(n_1-1)}{20}\Big)\|\theta_{n_0+1}\|^2 \\
&\leqslant (1 - \frac{\nu\lambda_N n_1}{40})\|\theta_{n_0}\|^2 \leqslant e^{-\frac{\nu\lambda_N n_1}{40}}\|\theta_{n_0}\|^2\,.
\end{aligned}
$$

Case II: $\|P_p\theta_{n_0+1}\|^2 \geqslant \frac{1}{4}\|\theta_{n_0+1}\|^2$ *(or equivalently* $\|P_c\theta_{n_0+1}\|^2 \leqslant \frac{3}{4}\|\theta_{n_0+1}\|^2$*).* By Lemma A.0.2, we have

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N U^m P_p\theta_{n_0+1}\|_1^2 \geqslant 5\lambda_N\|\theta_{n_0+1}\|^2\,, \tag{A.0.7}$$

and Lemma A.0.1 yields

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N U^m P_c\theta_{n_0+1}\|_1^2 \leqslant \frac{\lambda_N}{20}\|\theta_{n_0+1}\|^2\,. \tag{A.0.8}$$

Combining (A.0.7) and (A.0.7), we get

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N U^m \theta_{n_0+1}\|_1^2 \geqslant 2\lambda_N\|\theta_{n_0+1}\|^2\,. \tag{A.0.9}$$

By (2.1.26) and (2.1.21), we have

$$\begin{aligned}\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|\theta_{n_0+1+m}-\phi_{m+1}\|^2 &\leqslant \frac{n_1^2\nu}{n_1-1}\sum_{m=1}^{n_1-1}\|U\theta_{n_0+1+m}\|_1^2\\ &\leqslant \frac{n_1^2\nu}{n_1-1}\sum_{m=1}^{n_1-1}\|\nabla\varphi\|_{L^\infty}^{2m+2}\|\theta_{n_0+1}\|_1^2\\ &\leqslant \frac{n_1^2\nu\|\nabla\varphi\|_{L^\infty}^{2n_1+2}}{(n_1-1)(\|\nabla\varphi\|_{L^\infty}^2-1)}\|\theta_{n_0+1}\|_1^2\\ &\leqslant \frac{1}{4}\|\theta_{n_0+1}\|^2\,,\end{aligned}$$

which implies

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N(\theta_{n_0+1+m}-\phi_{m+1})\|_1^2 \leqslant \frac{\lambda_N}{4}\|\theta_{n_0+1}\|^2\,. \tag{A.0.10}$$

Equation (A.0.9) together with (A.0.10) gives

$$\sum_{m=1}^{n_1-1}\|\theta_{n_0+1+m}\|_1^2 \geqslant \sum_{m=1}^{n_1-1}\|P_N\theta_{n_0+1+m}\|_1^2 \geqslant \frac{\lambda_N}{2}(n_1-1)\|\theta_{n_0+1}\|^2\,. \tag{A.0.11}$$

We now use (2.1.21) again to get

$$\sum_{m=1}^{n_1-1}\mathcal{E}_\nu\theta_{n_0+m} \geqslant \lambda_N(n_1-1)\|\theta_{n_0+1}\|^2\,,$$

which, as before, yields

$$\|\theta_{n_0+n_1}\|^2 \leqslant e^{-\frac{\nu\lambda_N n_1}{2}}\|\theta_{n_0}\|^2 \,.$$

This proves (A.0.4) as desired, finishing the proof. $\square$

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