LECTURE NOTES ON STOCHASTIC CALCULUS FOR FINANCE
FALL 2021

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Contents

Note: The page numbers and links will not be correct in the annotated version.

2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2022-23/944-scalc-fina
- TA's: Jonghwa Park <jonghwap@andrew.cmu.edu>.
- Homework Due: 10:10AM Oct 27, Nov 3, 10, 22, 29, Dec 6
- Midterm: Tue, Nov 15, in class
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ One homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - – Not writing up solutions independently and/or plagiarizing solutions
    - – Turning in solutions you do not understand.
    - – Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 10% homework, 30% midterm, 60% final. + attendance

Course Outline.

- Review of Fundamentals: Replication, arbitrage free pricing.
- Quick study of the multi-period binomial model.
  - ▷ Simple example of replication / arbitrage free pricing.
  - ▷ Understand conditional expectations. (Have an explicit formula.)
  - ▷ Understand measurablity / adaptedness. (Can be stated easily in terms of coin tosses that have / have not occurred.)
  - ▷ Understand risk neutral measures. Explicit formula!
- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.

3. Replication and Arbitrage

3.1. Replication and arbitrage free pricing.

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ Mathematically: For any trading strategy such that $X_0 = 0$, and $X_n \geqslant 0$, you must also have $X_n = 0$ almost surely.
  - ▷ Equivalently: There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.

Stock

Bond

Market

& X_0 = Wealth at time $0 = 0$

X_n = " " " $n \geq 0$

then $X_n = 0$

- **Arbitrage free price**
  - ▷ Now consider a non-traded asset Y (e.g. an option). How do you price it?
  - ▷ *Arbitrage free price:* If given the opportunity to trade Y at price V_0, the market remains arbitrage free, then we say V_0 is the arbitrage free price of Y.

No options.

Stocks

Bank

Market (A. F.)

Want to trade a call option?

Call option at price $\boxed{V_0}$ → If market remains A F

then V_0 = A F P

- **Replication**
  - ▷ *We will almost always find the arbitrage free price by* ***replication.***
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Try and *replicate the payoff*:
    - – Start with X_0 dollars.
    - – Use only traded assets and ensure that at maturity $X_N = V_N$.
  - ▷ Then the arbitrage free price is uniquely determined, and must be X_0.

Remark 3.1. The arbitrage free price is *unique* if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.

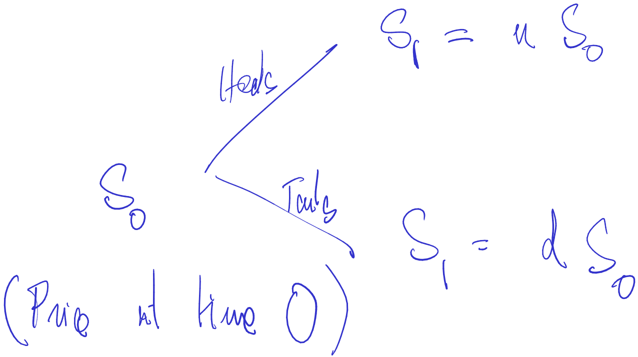
(FTAP).

3.2. **Example: One period Binomial model.**

- Consider a market with a stock, and money market account.
- Interest rate for borrowing and lending is r. No transaction costs. Can buy and sell fractional quantities of the stock.
- *Model assumption:* Flip a coin that lands heads with probability $p_1 \in (0,1)$ and tails with probability $q_1 = 1 - p_1$. Model $S_1 = uS_0$ if heads, and $S_1 = dS_0$ if tails.
  - ▷ S_0 is stock price at time 0 (known).
  - ▷ S_1 is stock price after one time period (random).
  - ▷ u, d are model parameters (pre-supposed). Called the up and down factors. (Will always assume $0 < d < u$.)

Proposition 3.2. *There's no arbitrage in this model if and only if* $d < 1 + r < u$.

Proof.



Check : Say $\boxed{\alpha = 1+r}$

Find arb :

At time 0 $\begin{cases} 1 \text{ share Stock} \\ -S_0 \quad \text{cash} \end{cases}$

$$X_0 = 0 = \underbrace{S_0}_{\text{1 share}} - \underbrace{S_0}_{\text{cash}}$$

$$X_1 = \underbrace{S_1}_{\text{new stock price}} - (1+r)S_0$$

$$= \begin{cases} \boxed{uS_0 - (1+r)S_0} & \text{if heads} \\ \boxed{dS_0 - (1+r)S_0} & \text{if tails} \end{cases}$$

$$\boxed{(u-(1+r))S_0} > 0$$

0

Proposition 3.3. *Say a security pays* $\boxed{V_1}$ *at time* 1 *(*V_1 *can depend on whether the coin flip is heads or tails). The arbitrage free price at time* 0 *is given by*

$$V_0 = \frac{1}{1+r}\big(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)\big) = \boxed{\frac{1}{1+r}\tilde{\boldsymbol{E}} V_1}\,, \quad \text{where } \tilde{p}_1 = \frac{1+r-d}{u-d}\,, \quad \tilde{q}_1 = \frac{u-(1+r)}{u-d}\,.$$

The replicating strategy holds $\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)S_0}$ *shares of stock at time* 0.

Proof.

Replicate the payoff by buying $\boxed{\Delta_0}$ shares at time 0

& starting with X_0 \$

$$X_0 = \underbrace{\Delta_0 S_0}_{\text{stock}} + \underbrace{\left(X_0 - \Delta_0 S_0\right)}_{\text{cash}}$$

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \overset{\text{Want}}{=} V_1$$

If heads: $X_1 = \Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0) = V_1(H)$ $\times \tilde{p}$

If tails: $X_1 = \Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0) = V_1(T)$ $\times \tilde{q}$

2 Eq & 2 unknowns $(X_0 \;\&\; \Delta_0)$ Solve.

Find $\tilde{p}$ & $\tilde{q}$ so that $\boxed{\tilde{p} + \tilde{q} = 1 \quad \& \quad u\tilde{p} + d\tilde{q} = (1+r)}$

$$\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T) = \Delta_0 S_0 \underbrace{(\tilde{p} u + \tilde{q} d)}_{1+r} + \underbrace{(\tilde{p} + \tilde{q})}_{1} (1+r)(X_0 - \Delta_0 S_0)$$

$$= (1+r)\cancel{\Delta_0 S_0} + (1+r)(X_0 - \cancel{\Delta_0 S_0})$$

$$\boxed{\Rightarrow X_0 = \frac{\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)}{(1+r)}}$$

Find Δ_0: Subtract the 2 eq $\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)\, S_0}$

Find $\tilde{p}, \tilde{q}$: $\tilde{p} = 1 - \tilde{q}$

$$u\tilde{p} + d\tilde{q} = 1+r \Rightarrow u\tilde{p} + d(1-\tilde{p}) = 1+r$$

$$\Rightarrow \tilde{p}(u-d) = 1+r-d$$

$$\Rightarrow \tilde{p} = \frac{1+r-d}{u-d} \quad \& \quad \tilde{q} = \frac{u-(1+r)}{d}$$

4. **Multi-Period Binomial Model.**

- Same setup as the one period case $0 < d < 1 + r < u$, and toss coins that land heads with probability p_1 and tails with probability q_1.
- Except now the security matures at time $N > 1$.
- Stock price: $S_{n+1} = uS_n$ if $n + 1$-th coin toss is heads, and $S_{n+1} = dS_n$ otherwise.
- To replicate it a security, we start with capital X_0.
- Buy Δ_0 shares of stock, and put the rest in cash.
- Get $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$.
- Repeat. *Self Financing Condition:* $X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$.
- *Adaptedness:* Δ_n can only depend on outcomes of coin tosses before n!

Proposition 4.1. *Consider a security that pays V_N at time N. Then for any $n \leqslant N$:*

$$\boxed{V_n = \frac{1}{(1+r)^{N-n}} \tilde{\boldsymbol{E}}_n V_N}\,, \qquad \boxed{\Delta_n = \frac{V_{n+1}(\omega_{n+1} = H) - V_{n+1}(\omega_{n+1} = T)}{(u-d)S_n}}\,.$$

- *V_n is the arbitrage free price at time $n \leqslant N$.*
- *Δ_n is the number of shares held in the replicating portfolio at time n (trading strategy).*

Question 4.2. *Why does this work?*

Question 4.3. *What is $\tilde{\boldsymbol{E}}_n$? (It's different from $\boldsymbol{E}$, and different from $\boldsymbol{E}_n$).*

$\tilde{E}_n$

$\sim$: Exp wrt the "Risk Neutral Measure"

$n \to$ "Conditional exp on $\mathcal{F}_n$" (IOU)

4.1. **Quick review probability (finite Sample spaces).** This is just a quick review for you to fix notation. You should already be familiar with this material from previous courses, and we won't go over it in class. We will, however, spend some time studying conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

ith

Definition 4.4. The *sample space* is the set $\Omega = \{(\omega_1, \ldots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss}\}$.

▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$. (Each ω_i could also represent the outcome of the roll of a M sided die.)

Definition 4.5. A *sample point* is a point $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$.

▷ Each sample point represents the outcome of a sequence of *all* coin tosses from 1 to N.

Definition 4.6. A *probability mass function* (PMF for short) is a function $p \colon \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Example 4.7. Typical example: Fix $p_1 \in (0, 1)$, $q_1 = 1 - p_1$ and set $p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$. Here $H(\omega)$ is the number of heads in the sequence $\omega = (\omega_1, \ldots, \omega_N)$, and $T(\omega)$ is the number of tails.

Definition 4.8. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

▷ $\boldsymbol{P}$ is called the probability measure associated with the PMF p.

Example 4.9. $A\{\omega \in \Omega \mid \omega_1 = +1\}$. Check $\boldsymbol{P}(A) = p_1$.

4.2. **Random Variables and Independence.**

Definition 4.10. A *random variable* is a function $X \colon \Omega \to \mathbb{R}$.

Example 4.11. $X(\omega) = \begin{cases} 1 & \omega_2 = +1\,, \\ -1 & \omega_2 = -1\,, \end{cases}$ is a random variable corresponding to the outcome of the second coin toss.

Definition 4.12. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 4.13. Note if $\text{Range}(X) = \{x_1, \dots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 4.14. The *variance* of a random variable is $\text{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 4.15. Note $\text{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

Definition 4.16. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 4.17. The events $A_1, \dots, A_n$ are independent if for any sub-collection $A_{i_1}, \dots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k})\,.$$

Remark 4.18. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Definition 4.19. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

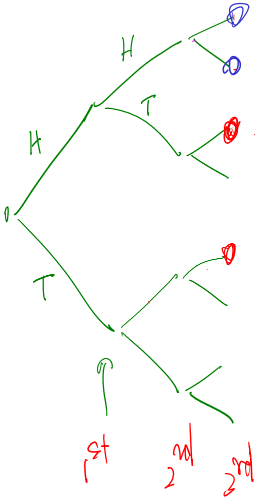
Definition 4.20. The random variables $X_1, \dots, X_n$ are independent if for all $x_1, \dots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \dots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 4.21. Independent random variables are uncorrelated, but not vice versa.

4.3. Filtrations.

- Let $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \in \pm 1 \text{ represents the outcome of a coin toss.}\}$.
  - ▷ It is convenient to visualize Ω and random variables by drawing trees.
  - ▷ E.g. X = outcome (± 1) of the second coin toss, Y = number of heads, etc.



	X	Y
	1	3
	1	2
(H, T, H)	−1	2
	−1	1
(T, H, H)	1	2
	1	1
	−1	1
	−1	0

$$X = \begin{cases} 1 & 2^{\text{nd}} \text{ toss} = H \\ -1 & 2^{\text{nd}} \text{ toss} = T \end{cases}$$

X is $\mathcal{F}_1$ - meas NO

X is $\mathcal{F}_2$ - meas Yes

X is $\mathcal{F}_2$ meas Yes

Y is $\mathcal{F}_3$ meas.

Definition 4.22. We define a *filtration* on Ω as follows:

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- $\mathcal{F}_1$ = all events that can be described by $\boxed{\text{only the first coin toss.}}$ E.g. $\boxed{A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_1.}$
- $\mathcal{F}_2$ = all events that can be described by only the first two coin toss.
  - E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_2$, $B = \{\omega \mid \omega_1 = +1, \omega_2 = -1\} \in \mathcal{F}_2$.
- $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses.
  - E.g. $A = \{\omega \mid \omega_1 = 1, \omega_3 = -1, \omega_n = 1\} \in \mathcal{F}_n$.

Remark 4.23. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Remark 4.24. If $A, B \in \mathcal{F}_n$, then so do A^c, B^c, $A \cap B$, $A \cup B$, $A - B$, $B - A$.

$$F_1 = \{\emptyset, \Omega, \overbrace{\{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}}, \ldots\}$$

Definition 4.25. Let $n \in \{0, \ldots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \ldots, \omega_n$. (and not $\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N$)

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Remark 4.26 (Use in Finance). For every n, the trading strategy at time n (denoted by Δ_n) must be $\mathcal{F}_n$ measurable. We can not trade today based on tomorrows price.

Example 4.27. If we represent Ω as a tree, $\mathcal{F}_n$ measurablity can be visualized by checking constancy on leaves.

4.4. Conditional expectation.

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \operatorname{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega)) \leftarrow \mathcal{F}_n\text{- meas.}$$

$$\text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\} \in \mathcal{F}_n$$

Remark 4.29. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boxed{\boldsymbol{E}_n X}$ is the "best approximation" of X given only the first n coin tosses.

$$EX = \sum x_i P(X = x_i)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proposition 4.32. *The conditional expectation* $\boxed{\boldsymbol{E}_n X}$ *defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\boxed{\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega).}$

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X$.

$\boxed{A \in \mathcal{F}_n,}$

Avg over A of $X = \frac{1}{P(A)} \boxed{\sum_{\omega \in A} p(\omega) X(\omega)}$

$\|$

Avg " A of $E_n X = \frac{1}{P(A)} \boxed{\sum_{\omega \in A} p(\omega) E_n X(\omega)}$

Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y$.

$X \longrightarrow \mathcal{F}_n$ - meas,

$E_n X = X$

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boldsymbol{E}_n X = X$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $A \in \mathcal{F}_n$ and $B \subseteq \mathbb{R}$, the events A and $\{X \in B\}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, ..., n^{th} coin tosses and *not* the 1^{st}, 2^{nd}, ..., n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

Proposition 4.39 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

Last time:

1 No arb

2 $\boxed{\text{AFP}}$

Stock

Bank

NTA (option)

V_0

Market

3 Compute AFP by replication

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$$\boxed{\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \operatorname{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega))}$$

where $$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega_1' = \omega_1, \ldots, \omega_n' = \omega_n\}$$

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(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$. $\cancel{\frac{1}{P(A)}}$

$P(A)$

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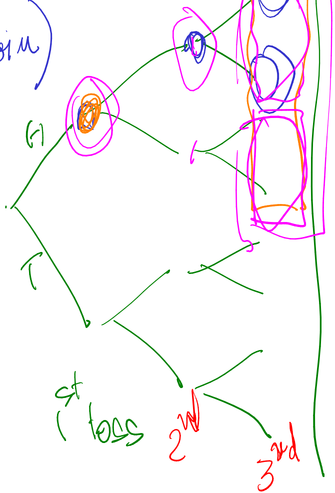
(2) (Tower property) *If $m \leqslant n$, then* $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.

(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boxed{\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y}$.

Eg :

(Fair coin)



X	$E_3 X$	$E_2 X$	$E_1 X$	$E_1(E_2 X)$
1	1	1.5	2.5	2.5
2	2	1.5	2.5	
3	3	3.5	2.5	
4	4	3.5	2.5	
5	5	.		
6	6	.		
7	7	.		
8	8	.		

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boxed{\boldsymbol{E}_n X = X}$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $\boxed{A \in \mathcal{F}_n}$ and $\boxed{B \subseteq \mathbb{R}}$, the events A and $\boxed{\{X \in B\}}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, $\dots$, n^{th} coin tosses and *not* the 1^{st}, 2^{nd}, $\dots$, n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

$\boxed{X}$ is ind of the 1^{st} n coin tosses

$$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}$$

Proposition 4.39 (Independence lemma). *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \ldots, x_m\} = X(\Omega)\,.$$

$\mathcal{F}_n$ ind

$\mathcal{F}_n$ - meas

$$\boxed{E f(X) = \sum_{x_i \in \text{Range}(X)} f(x_i) P(X = x_i)}$$

Eg: $f(x,y) = \sqrt{X^2 + Y^2}$

X ind of $\mathcal{F}_n$

Y meas wrt $\mathcal{F}_n$

4.5. **Martingales.**

Definition 4.40. A *stochastic process* is a collection of random variables X_0, X_1, ..., X_N.

Example 4.41. Typically X_n is the wealth of an investor at time n, or S_n is the price of a stock at time n.

Definition 4.42. A stochastic process is $\boxed{\textit{adapted}}$ if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Remark 4.43. Requiring processes to be adapted is fundamental to Finance. Intuitively, being adapted forbids you from trading today based on tomorrows stock price. All processes we consider (prices, wealth, trading strategies) will be adapted.

Example 4.44 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 4.45 (Stock price). Let $S_0 \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

$S_0 > 0$

adapted stochastic process.

Definition 4.46. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 4.47. Intuition: A martingale is a "fair game".

Example 4.48 (Unbiased random walk). If $\xi_1, \dots, \xi_N$ are i.i.d. and mean zero, then $X_n = \sum_{k=1}^n \xi_k$ is a martingale.

$X_0 = 0$

$X_1 = X_0 + \xi_1$

$X_2 = X_1 + \xi_2$

$X_{n+1} = X_n + \xi_{n+1}$

Assume $E\xi_n = 0$ & ξ_{n+1} is ind of ξ_n.

Then X_n is a mg:

$E_n X_{n+1} \overset{\text{Want}}{=} X_n$

u

$$
\begin{aligned}
E_n(X_{n+1}) &= E_n(X_n + \xi_{n+1}) \\
&= E_n X_n + E_n \xi_{n+1} \\
&= X_n + E\xi_{n+1} \quad \left(\because X_n \text{ is } \mathcal{F}_n\text{-meas} \atop \xi_{n+1} \text{ is } \mathcal{F}_n\text{-ind} \right) \\
&= X_n
\end{aligned}
$$

Remark 4.49. If M is a martingale, then for every $m \leqslant n$, we must have $\boxed{\boldsymbol{E}_m M_n = M_m}$.

Remark 4.50. If M is a martingale then $\boxed{\boldsymbol{E} M_n} = \boldsymbol{E} M_0 = M_0$.

$$M_n = E_n M_{n+1}$$

$$M_{n-1} = E_{n-1} M_n = E_{n-1}\left(E_n M_{n+1}\right) = \boxed{E_{n-1} M_{n+1}}$$

Tower

$$M_{n-2} = E_{n-2} M_{n-1} = E_{n-2} E_{n-1} M_{n+1} = E_{n-2} M_{n+1}.$$

etc.

$$E M_n = E\left(E_0 M_n\right) \overset{mg}{=} E M_0 = M_0$$

($\because$ M_0 is $\mathcal{F}_0$ meas.

M_0 doesn't dep on any coin tosses)

4.6. **Change of measure.**

- Gambling in a Casino: If it's a martingale, then on average you won't make or lose money.
- Stock market: Bank always pays interest! Not looking for a "break even" strategy.
- Mathematical tool that helps us price securities: Find a *Risk Neutral Measure.*
  - ▷ Discounted stock price is (usually) not a martingale.
  - ▷ Invent a "risk neutral measure" which the discounted stock price is a martingale.
  - ▷ Securities can be priced by taking a conditional expectation *with respect to the risk neutral measure.* (That's the meaning of $\tilde{\boldsymbol{E}}_n$ in Proposition 4.1.)

$V_N \to$ Payoff at time N

$V_n =$ AFP at time n

$D_n = \frac{1}{(1+r)^n}$ discount factor

RNP Formula

$$\boxed{V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)}$$

Definition 4.51. Let $D_n = (1+r)^{-n}$ be the discount factor. (So $D_n\$$ in the bank at time 0 becomes $1\$$ in the bank at time n.)

- Invent a new probability mass function $\tilde{p}$.
- Use a tilde to distinguish between the new, invented, probability measure and the old one.
  - ▷ $\tilde{\boldsymbol{P}}$ the probability measure obtained from the PMF $\tilde{p}$ (i.e. $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$).
  - ▷ $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ conditional expectation with respect to $\tilde{\boldsymbol{P}}$ (the new "risk neutral" coin)

Definition 4.52. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Definition 4.53. A *risk neutral measure* is an equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale. (I.e $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = D_n S_n$.)

Remark 4.54. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 4.55. Proposition 4.1 says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n. (Called the risk neutral pricing formula.)

Proposition 4.56. *Let $\tilde{\boldsymbol{P}}$ be an equivalent measure under which the coins are i.i.d. and land heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$.*

(1) *Under $\tilde{\boldsymbol{P}}$, we have $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = \frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} D_n S_n$.*

(2) *$\tilde{\boldsymbol{P}}$ is the risk neutral measure if and only if $\tilde{p}_1 u + \tilde{q}_1 d = 1 + r$. (Explicitly $\tilde{p}_1 = \frac{1+r-d}{u-d}$, and $\tilde{q}_1 = \frac{u-(1+r)}{u-d}$.)*

(1) Compute $\tilde{E}_n(D_{n+1} S_{n+1})$:

Let $X_{n+1} = \begin{cases} u & \text{if } n+1^{\text{th}} \text{ coin is heads} \\ d & \text{if } n+1^{\text{th}} \text{ coin is tails} \end{cases}$

Note $S_{n+1} = S_n X_{n+1}$

$$\Rightarrow \tilde{E}_n(D_{n+1} S_{n+1}) = D_{n+1} \tilde{E}_n(S_{n+1})$$

$$= D_{n+1}\,\tilde{E}_n\left(S_n X_{n+1}\right)$$

$$= D_{n+1} S_n\,\tilde{E}_n X_{n+1} \qquad \left(\because S_n \text{ is } \mathcal{F}_n \text{ meas}\right)$$

$$= \frac{D_n}{1+r} S_n\,\tilde{E} X_{n+1} \qquad \left(\because X_{n+1} \text{ is indep of } \mathcal{F}_n\right)$$

$$= D_n S_n \left(\frac{u\tilde{p}_1 + d\tilde{q}_1}{1+r}\right)$$

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some $\boxed{\textit{adapted}}$ process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Check: If X_n is self fin (Know $D_n S_n$ is a $\tilde{P}$ mg)
Then $(D_n X_n)$ is a $\tilde{P}$ - mg.

Pf: Know: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

Want $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$

$$\tilde{E}_n\left(D_{n+1}X_{n+1}\right) = \tilde{E}_n\left(D_{n+1}\Delta_n S_{n+1} + D_{n+1}(1+r)(X_n - \Delta_n S_n)\right)$$

$$= \Delta_n \underbrace{\tilde{E}_n\left(D_{n+1}S_{n+1}\right)} + \cancel{\tilde{E}_n} \underbrace{D_n\left(X_n - \Delta_n S_n\right)}$$

$\tilde{E}_n$ — meas.

$$= \cancel{\Delta_n D_n S_n} + D_n\left(X_n - \cancel{\Delta_n S_n}\right) = D_n X_n$$

Last time: RNM

Equivalent meas $\tilde{P}$ under which

$$\tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$$

(Disc stock is a $\tilde{P}$-mg).

Binom model:

Coin flip $\xrightarrow{p_1}$ Heads, $\xrightarrow{q_1}$ tails

RNM:

Coin flip: Heads $\rightarrow \tilde{p}_1$, tails $\rightarrow \tilde{q}_1$

$$\tilde{p}_1 = \frac{(1+r) - d}{u - d}$$

Last time

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)}$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Proof of Proposition 4.1.

Security Pays V_N at time N

AFP at time $n \leq N$ is

RNP formula

$$\boxed{V_n = \frac{1}{D_n} \tilde{E}_n \left(D_N V_N \right)}$$

D_n = Discount factor $= \dfrac{1}{(1+r)^n}$

Pf: Want a R. Pf.

$\iff$ Want a self fin port $\Rightarrow$ $\boxed{X_N = V_N}$.

$\Updownarrow$

$\boxed{\text{Disc wealth is a } \tilde{P}\text{-mg.}}$

Choose $X_n = \dfrac{1}{D_n} \tilde{E}_n(D_N X_N)$ & $\boxed{X_N = V_N}$.

Know $X_N = V_N$ (by choice)

$\boxed{\Rightarrow D_{n+1} X_{n+1} = \tilde{E}_{n+1}(D_N X_N)}$

NTS $D_n X_n$ is a $\tilde{P}$ - mg

$\hookrightarrow$ Check: $D_n X_n = \tilde{E}_n(D_{n+1} X_{n+1})$

Compute $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n(\tilde{E}_{n+1}(D_N X_N))$

Tower $= \tilde{E}_n(D_N X_N)$

$= D_n X_n$!! Done!!

Example 4.59. Consider two stocks S^1 and S^2, $u = 2$, $d = 1/2$.

- ▷ The coin flips for S^1 are heads with probability 90%, and tails with probability 10%.
- ▷ The coin flips for S^2 are heads with probability 99%, and tails with probability 1%.
- ▷ Which stock do you like more?
- ▷ Amongst a call option for the two stocks with strike K and maturity N, which one will be priced higher?

Both call options are worth exactly the same

[RNM only depends on u, d, r & N
does not depend on p & q]

Remark 4.60. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

$\to$ $\tilde p_1$ & $\tilde q_1$ Do not depend on p_1 & q_1

Concepts that will be generalized to continuous time.

- Probability measure: Lebesgue integral, and not a finite sum. Same properties.
- Filtration: Same intuition. No easy description.
- Conditional expectation: Same properties, no formula.
- Risk neutral measure: Formula for $\tilde{\boldsymbol{P}}$ is complicated (Girsanov theorem.)
- Everything still works because of of Theorem 4.57. Understanding why is harder.

Self fin $\iff$ Disc wealth is a $\tilde P$ mg

5. Stochastic Processes

5.1. Brownian motion.

$\mathrm{Var}(\xi_i) = 1.$

Pollen

- Discrete time: Simple Random Walk.
  - ▷ $\boxed{X_n = \sum_1^n \xi_i,}$ where ξ_i's are i.i.d. $\boxed{\boldsymbol{E}\xi_i = 0,}$ and $\boxed{\mathrm{Range}(\xi_i) = \{\pm 1\}.}$
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon}\, Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\mathrm{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $\boxed{W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon.}$

Weiner Process

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.

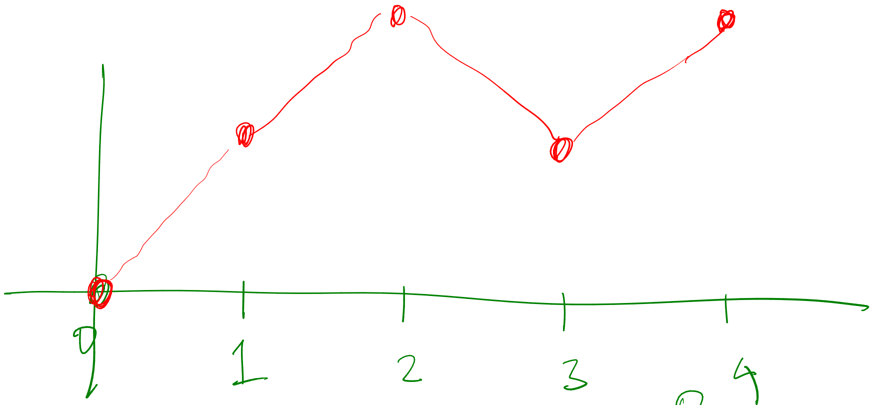
$X_0 = 0$

Outcome of k^{th} play is the RV $\boxed{\xi_k}$

Cumu wealth

$X_1 = X_0 + \xi_1$

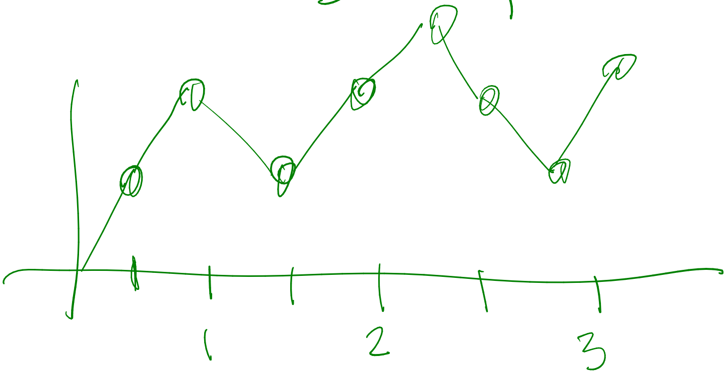
$X_{n+1} = X_n + \xi_{n+1}$



time

Speed up time

k ,



$$\mathrm{Var}(X_n) = \mathrm{Var}\left(\sum_{1}^{n} \xi_k\right) = \sum_{1}^{n} \mathrm{Var}(\xi_k) \quad \text{(indep)}$$

$$= n.$$

$$\boxed{Y^{\varepsilon}_t = \sqrt{\varepsilon}\, Y_{t/\varepsilon}}$$

(Say t is an int mult of ε)

$$\mathrm{Var}(Y^{\varepsilon}_t) = \varepsilon\, \mathrm{Var}(Y_{t/\varepsilon}) = \cancel{\varepsilon}\ \frac{t}{\cancel{\varepsilon}}$$

- If t, s are multiples of ε: $\boxed{Y_t^\varepsilon - Y_s^\varepsilon} \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \boxed{\mathcal{N}(0, t-s)}$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a $\boxed{\textit{continuous process}}$ such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

$\boxed{E\xi_i = 0, \ E\xi_i^2 = 1}$

(1)

$\frac{1}{N} \sum_1^N \xi_i \xrightarrow{N \to \infty} 0 \ (LLN)$

CLT $\frac{1}{\sqrt{N}} \sum_1^N \xi_i \xrightarrow{N \to \infty} N(0,1)$

Say t & s are mult of ε. ($s = m\varepsilon$ & $t = n\varepsilon$.)

Then $\quad Y^{\varepsilon}_t - Y^{\varepsilon}_s = \sqrt{\varepsilon} \sum_{m+1}^{n} \xi_k$

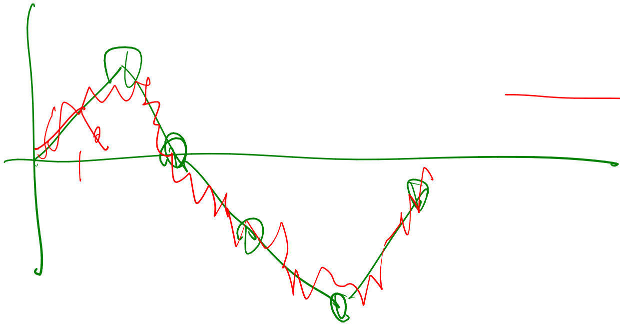
$$= \sqrt{\varepsilon} \cdot \Big(\text{sum of } \underbrace{\left(\frac{t-s}{\varepsilon}\right)}_{N} \text{ iid RV's} \Big)$$

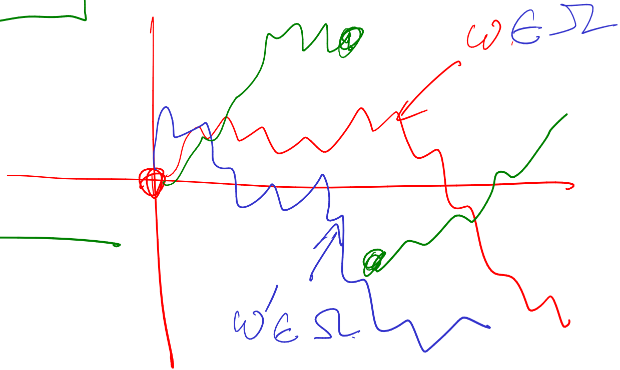
$$= \sqrt{t-s}\, \underbrace{\boxed{\frac{1}{\sqrt{N}} \left(\text{sum of } N \text{ iid RV's}\right)}}_{\downarrow\; N(0,1)} \qquad \left| \; \frac{t-s}{\varepsilon} = N \right.$$

$$\sim N(0, t-s).$$

5.2. **Sample space, measure, and filtration.**

- Discrete time: Sample space $\Omega = \{(\omega_1, \dots, \omega_N) \mid \omega_i$ represents the outcome of the i^{th} coin toss$\}$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.





- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} : \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\left(\bigcup_1^\infty A_n\right) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$. $(\& \; A_n \subseteq A_{n+1})$
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

$X : \Omega \longrightarrow \mathbb{R}$

Given any $\omega \in \Omega$, $X(\omega) \in \mathbb{R}$.

Is $X > 0$?

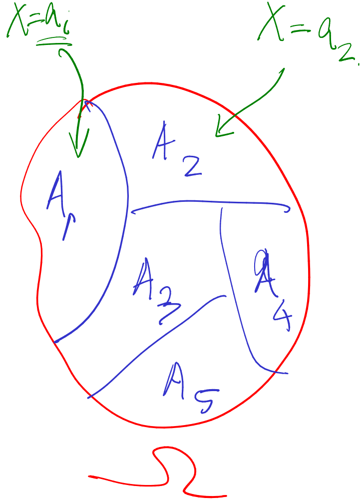
$\{X > 0\} = \{\omega \mid X(\omega) > 0\}$

Q: $P(X > 0)$

- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_\Omega X \, d\boldsymbol{P} = \int_\Omega X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$. $= \sum a_i P(X = a_i)$
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

Discrete rv

$$\boxed{EX = \sum_{x_i \in Ran(X)} x_i \, P(X = x_i)}$$



Proposition 5.3 (Useful properties of expectation).

α, β not random.

(1) (Linearity) $\alpha, \beta \in \mathbb{R}$, X, Y *random variables,* $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E} X + \beta \boldsymbol{E} Y$.

(2) (Positivity) *If* $X \geqslant 0$ *then* $\boldsymbol{E} X \geqslant 0$. *If* $X \geqslant 0$ *and* $\boldsymbol{E} X = 0$ *then* $X = 0$ *almost surely.*

(3) (Layer Cake) *If* $X \geqslant 0$, $\boldsymbol{E} X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt$.

(4) *More generally, if* φ *is increasing,* $\varphi(0) = 0$ *then* $\boldsymbol{E} \varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt$.

(5) (Unconscious Statistician Formula) *If PDF of* X *is* p, *then* $\boldsymbol{E} f(X) = \int_{-\infty}^\infty f(x) p(x)\, dx$.

lazy

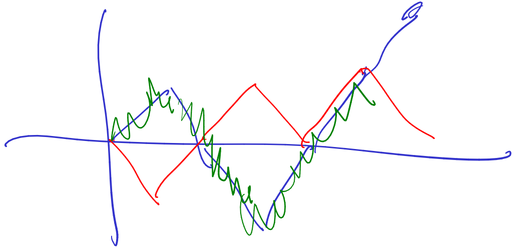
$\big(X = 0$ a.s. means $P(X=0) = 1\big)$.

φ' = derivative of φ

PDF of X is p, then $E f(X) = \int_{\mathbb{R}} f(x)\, p(x)\, dx$

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\mathcal{F}_t$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0, 1\}\}$.

$X_{n+1} = X_n + \xi_{n+1}$ (ξ_{n+1} = outcome of $n+1$th coin toss)



Eg: Say $s < t$.

$\{W_s > 0\} \in \mathcal{F}_t$

$\{W_s \in [1,2] \,\&\, W_t < 0\} \in \mathcal{F}_t$

$$\{W_{t+1} > 0\} \notin \mathcal{F}_t$$

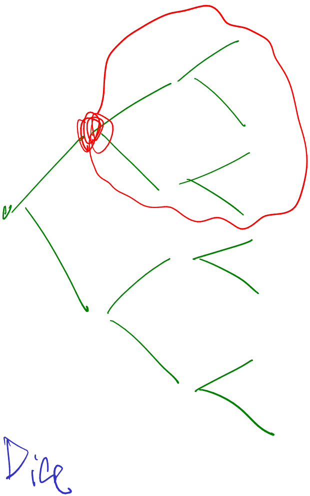
$$\{W_{t+1} > 0\} \in \mathcal{F}_{t+1}$$

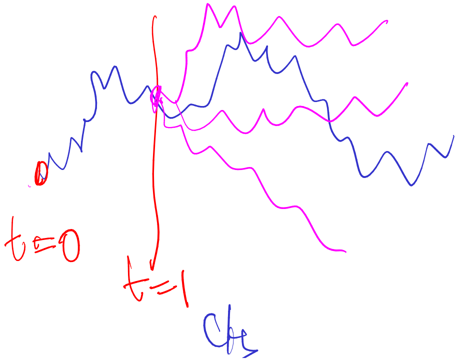
5.3. **Conditional expectation.**

- Notation $\boxed{\boldsymbol{E}_t(X)} = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega'(s) = \omega(s)\ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

$E_1 X$

avg over all cointosses after 1st one.





Definition 5.4. $\boldsymbol{E}_t X$ is the unique $\boxed{\textit{random variable}}$ such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

Remark 5.5. Choosing $A = \Omega$ implies $\boxed{\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X.}$

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

Lemma 5.8 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(\cancel{Y}X)\,, \qquad \textit{where} \qquad \cancel{g(y) = \boldsymbol{E} f(X, y)}\,.$$

Remark 5.9. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X, Y) = \int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$.

$$g(x) = E\left[f(x, Y)\right]$$

subst $x = X$

$g(X) =$ "Avg $f(X, Y)$ in Y & leave X alone"

5.4. **Martingales.**

Definition 5.10. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.11. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

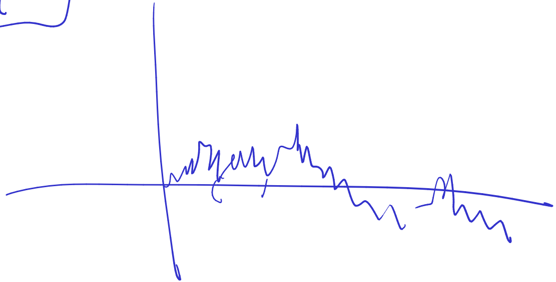
Proposition 5.12. *Brownian motion is a martingale.*

Proof.

Reminder: O.H. Today 3:30 & Tomorrow 12:00 (Zoom)

Last time: $\boxed{\text{Brownian Motion}}$

Cts time RW



W_t = BM at time t

① $W_t - W_s \sim N(0, t-s)$

② $W_t - W_s$ is <u>indep</u> of $\mathcal{F}_s$

5.4. **Martingales.**

Definition 5.11. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.12. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Disc time:

$$E_n M_{n+1} = M_n$$

Proposition 5.13. *Brownian motion is a martingale.*

Proof. Want to show $E_s W_t = W_s$

$$E_s W_t = E_s(W_t - W_s + W_s) = \underbrace{E_s(W_t - W_s)}_{\text{Indep}} + \underbrace{E_s W_s}_{\mathcal{F}_s \text{ meas}}$$

$$= \underbrace{E(W_t - W_s)}_{0} + W_s$$

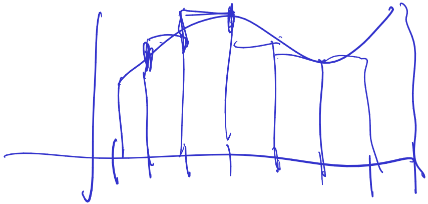
$$= W_s.$$

6. **Stochastic Integration**

Partition

6.1. **Motivation.**

- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$
- Net gain/loss from changes in stock price: $\displaystyle\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  - ▷ $\|P\| = \max_k (t_{k+1} - t_k)$. ← Mesh size of P
  - ▷ Riemann-Stieltjes integral: $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t,$
  - ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  - ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.



6.2. **First Variation.**

$\Sigma (X_{t_{k+1}} - X_{t_k}) = X_{t_n} - X_0$

Definition 6.1. For any process X, define the $\boxed{\textit{first variation}}$ by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|\,.$$

Remark 6.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]}X < \infty$.

Proposition 6.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

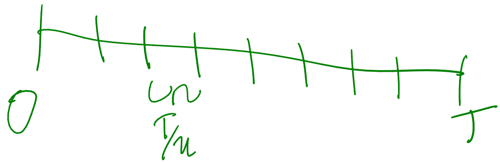
Remark 6.4. In fact, $V_{[0,T]}W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 6.5. The Riemann-Stieltjes integral $\int_0^T b_t\, dW_t$ *does not exist.*

$\Delta_k W = W_{t_{k+1}} - W_{t_k}$

$$\to E\, V_{[0,T]} W = \lim_{\|P\|\to 0} E \sum |\Delta_k W|$$

Say $P \to$ "uniform"



I.e. say $t_{k+1} - t_k = \frac{T}{n}$

$$E|\Delta_k W| = E\left|W_{t_{k+1}} - W_{t_k}\right| = E\left|N(0, t_{k+1} - t_k)\right|$$

$$= E\left|N\left(0, \frac{T}{n}\right)\right| = E\left|\sqrt{\frac{T}{n}}\, N(0,1)\right|$$

$$= \sqrt{\frac{T}{n}}\ \underbrace{E\left|N(0,1)\right|}_{\text{some finite constant.}}$$

ต่อ $\quad E\, V_{[0,T]} W = \lim\limits_{\|P\| \to 0} \sum\limits_{k=0}^{n-1} \underbrace{E\,|\Delta_k W|}$

$$= \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} \sqrt{\frac{T}{n}}\; E\,|N(0,1)|$$

$$= \lim_{\|P\| \to 0} \sqrt{nT}\; E\,|N(0,1)| \longrightarrow \infty$$

6.3. **Quadratic Variation.**

$V_T M = \lim_{\|P\| \to 0} \sum |M_{t_{k+1}} - M_{t_k}|$

Definition 6.6. If M is a continuous time adapted process, define

$$[M, M]_T = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 \,.$$

Proposition 6.7. *For continuous processes the following hold:*

1. *Finite first variation implies the* $\boxed{\textit{quadratic variation is } 0}$
2. *Finite (non-zero) quadratic variation implies the first variation is infinite.*

Revisit this later (Important)

Proposition 6.8. $[W,W]_T = T$ *almost surely.*

Remark 6.9. For use in the proof: $\operatorname{Var}(\mathcal{N}(0,\sigma^2)^2) = \boldsymbol{E}\mathcal{N}(0,\sigma^2)^4 - (\boldsymbol{E}\mathcal{N}(0,\sigma^2)^2)^2 = 2\sigma^4$.

$$3\sigma^4 - \sigma^4$$

$$\boxed{E\,N(0,\sigma^2)^2 = \sigma^2}$$

Proof:.

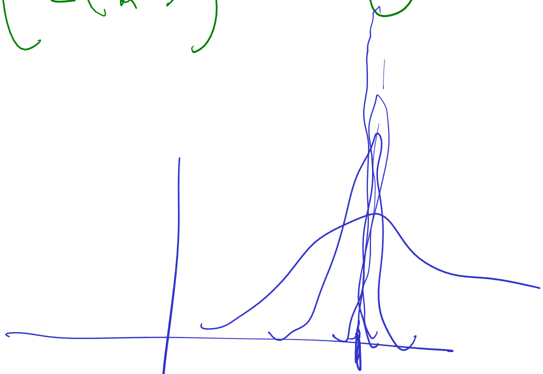
$$[W,W]_T = \lim_{\|P\|\to 0} \sum (\Delta_k W)^2$$

Assume uniform mesh

NTS $= T$ a.s.

Will show ① $\lim_{\|P\|\to 0} E\sum(\Delta_k W)^2 = T$

& ② $\lim_{\|P\|\to 0} \mathrm{Var}\left(\sum(\Delta_k W)^2\right) = 0$



T

Check ①:

$$E\sum(\Delta_k W)^2 = \sum E\, N(0, t_{k+1}-t_k)^2$$

$$= \sum (t_{k+1} - t_k) = T$$

---

(2) $$\mathrm{Var}\left(\sum_{k=0}^{n-1} (\Delta_k W)^2\right) = \sum_{k=0}^{n-1} \mathrm{Var}\left((\Delta_k W)^2\right) \quad (\text{by indep})$$

$$= \sum_{k=0}^{n-1} \mathrm{Var}\left(N(0, t_{k+1} - t_k)^2 \right)$$

$$= \sum_{k=0}^{n-1} \mathrm{Var}\left(N\left(0, \frac{T}{n}\right)^2 \right) = \sum_{k=0}^{n-1} 2 \frac{T^2}{n^2}$$

$$= \frac{2T^2}{n} \xrightarrow[(n \to \infty).]{\|P\| \to 0} 0$$

$$\boxed{\therefore\ [W,W]_t = t \quad \text{a.s.}}$$

Proposition 6.10. *$W_t^2 - [W,W]_t$ is a martingale.*

Check: $E_s\left(W_t^2 - [W,W]_t\right) \overset{\text{Want}}{=} W_s^2 - [W,W]_s$

$$E_s\left(W_t^2 - [W,W]_t\right) = E_s\left(W_t^2 - t\right)$$

$$= E_s W_t^2 - t$$

$$= E_s\left(W_t - W_s + W_s\right)^2 - t$$

$$= E_s\left[(W_t - W_s)^2 + W_s^2 + 2(W_t - W_s)W_s\right] - t$$

$$= E(W_t - W_s)^2 + W_s^2 + 2\,E_s\overbrace{\left[(W_t - W_s)W_s\right]}^{E_s \text{ meas}} - t$$

$$(W_t - W_s \sim N(0, t-s))$$

$$= \cancel{t} - s + W_s^2 + 2W_s \underbrace{E_s(W_t - W_s)}_{0} - \cancel{t}$$

$$= W_s^2 - s = W_s^2 - [W,W]_s.$$

$$\boxed{\Rightarrow W^2 - [W,W] \text{ is a } M_g.}$$

Theorem 6.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E}M_t^2 < \infty$ if and only if $\boldsymbol{E}[M,M]_t < \infty$.

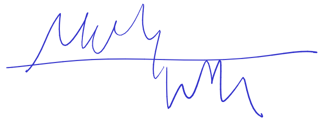
(2) In this case $M_t^2 - [M,M]_t$ is a continuous martingale.

(3) Conversely, if $M_t^2 - A_t$ is a martingale for any continuous, increasing process A such that $A_0 = 0$, then we must have $A_t = [M,M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

Remark 6.13. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 6.14. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.



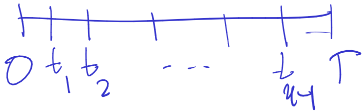
Say X is diff.

$$X_{t+h} - X_t \approx \left(\frac{dX}{dt}\right) \cdot h$$

Finite QV: $|X_{t+\delta t} - X_t|^2 \approx \delta t$

6.4. **Itô Integrals.**

- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i t_{i+1} - t_i$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$



$I_P(T) =$ "Itô sum"

Definition 6.15. The *Itô Integral* of D with respect to Brownian motion is defined by

$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) \, .$$

Remark 6.16. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

$\int_0^T D_t^2\,dt \to$ R. int

Theorem 6.17. *If* $\boldsymbol{E}\int_0^T D_t^2\,dt < \infty$ *a.s., then:*

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E}I(T)^2 < \infty$.

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I,I]_T = \int_0^T D_t^2\,dt$ *a.s.*

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

$$I_T = \int_0^T D_t\,dW_t$$

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\underbrace{\Big(\int_0^T D_t\,dW_t\Big)}_{\text{Ito int.}}^2 = \boldsymbol{E}\underbrace{\int_0^T D_t^2\,dt}_{\text{Riemann Int}}$

Proof.

Intuition ① $I_T = \int_0^T D_s\,dW_s$

Thm $\Rightarrow$ $\underline{I}$ is a mg & $[I,I]_T = \int_0^T D_s^2\,ds.$

② Know $I_t^2 - [I,I]_t$ is a mg.

3 $\Rightarrow$ $E\left(I_t^2 - [I,I]_t\right) = E_0\left(I_t^2 - [I,I]_t\right)$

$= I_0^2 - [I,I]_0 = 0$

4 $\Rightarrow$ $E I_t^2 = E[I,I]_t$

$= E\int_0^t D_s^2\,ds$ (by 1)

$\Rightarrow$ Itô ~~iso~~ isom.

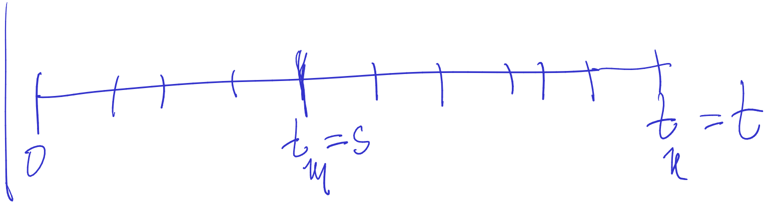
Intuition why Itô int is a mg?

Simplest case: Check $I_p(t)$ is a mg in a simple case.

Compute $E_s I_p(t) \overset{\text{Want}}{=} I_p(s)$

Say $s = t_m$

$t = t_n$ $\quad m < n$



$$I_p(s) = \sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$E_s I_p(t) = E_s \sum_{k=0}^{n-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) \qquad (s = t_m$$

$$= E_s \underbrace{\sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)}_{E_s \text{ meas } (s = t_m)} + E_s \sum_{k=m}^{n-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$=$$

$$= \underbrace{\sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)}_{I_p(s)} + \sum_{k=m}^{n-1} E_s D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s E_{t_k} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) \quad (\text{tower})$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s\left[D_{t_k} E_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)\right] \quad \left(\because D_{t_k} \text{ is } \mathcal{F}_{t_k} \text{ meas}\right)$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s\left[D_{t_k} \underbrace{E\left(W_{t_{k+1}} - W_{t_k}\right)}_{0} \right] \quad (\text{indep})$$

$$= I_p(s) \qquad \boxed{\Rightarrow \quad E_s I_p(t) = I_p(s)}$$

Last time

Goal → B.S formula (Price sec in c/s time)

Ito integrals & QV → Ito formula → Goal

Ito formula → Girsanov Thm → RNM → (better)

lim of Ito sums

$P = \{t_0 = 0 < t_1 \cdots < t_n = T\}$

$$\lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \Delta_{t_i} \left(W_{t_{i+1}} - W_{t_i}\right) = \int_0^T \Delta_s \, dW_s$$

Always assume $D \rightarrow$ adapted process

Notation reminder:

$$\int_0^T D_t \, dt = \text{Rieman Int}$$

$$\int_0^T D_t \, dW_t = \text{Itô int}$$

Theorem 6.17. *If* $\boxed{\boldsymbol{E}\int_0^T D_t^2\,dt < \infty}$ ~~*a.s.*~~*, then:*

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E}I(T)^2 < \infty$.

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I,I]_T = \int_0^T D_t^2\,dt$ *a.s.*

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\Big(\underbrace{\int_0^T D_t\,dW_t}_{\text{Itô int.}}\Big)^2 = \boldsymbol{E}\int_0^T \underbrace{D_t^2\,dt}_{\text{R-int}}$

Proof. Last time

Caution $EX^2 = E(X^2)$ (not $(EX)^2$)

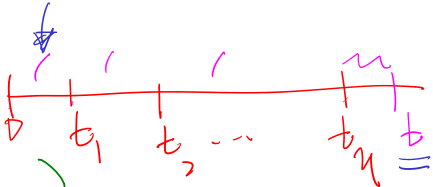
$\int_0^t D_s\,dW_s$ is a mg with qv $\int_0^t D_s^2\,ds$

$\Rightarrow \left(\int_0^t D_s\,dW_s\right)^2 - \int_0^t D_s^2\,ds$ is a mg & take E.

Intuition for Theorem 6.17 (2). Check $I_P(T)$ is a martingale.

Last time

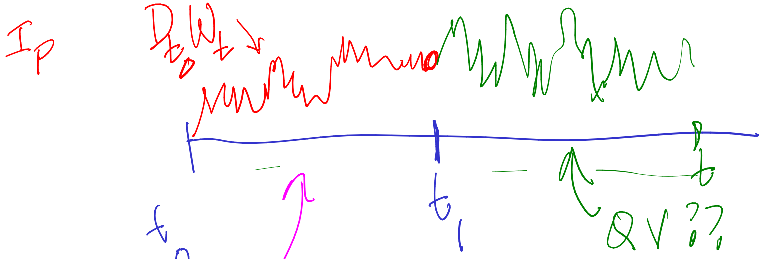
Why is $[I,I]_T = \int_0^T D_s^2 \, ds$



$$I_P(t) = \sum_{0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_t - W_{t_n})$$

$[I_P, I_P]$

1. Say $0 \le t < t_1$: $I_P(t) = D_{t_0}(W_t - \cancel{W_0})$



$$t \in [t_1, t_2]$$

$$I_p(t) = D_{t_0}(W_{t_1} - W_{t_0}) + D_{t_1}(W_t - W_{t_1})$$

indep

$$[I_p, I_p]_t = D_{t_0}^2 \, t$$

$$[I_p, I_p]_{t_1} = D_{t_0}^2 \, t_1$$

$$I_p(t) = D_{t_0}^2 (t_1 - t_0) + D_{t_1}^2 (t - t_1)$$

Say $t \in [t_n, t_{n+1})$

Expect $\quad [I_P, I_P]_t = \underbrace{\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (t - t_n)}$

$$\Big\downarrow \; \|P\| \to 0$$

$$\boxed{\int_0^t D_s^2 \, ds} \leftarrow QV \text{ of } \int_0^t D_s \, dW_s$$

Proposition 6.20. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes* (α, α̃ constants, not random)

$$\int_0^T (\alpha D_s + \tilde{\alpha}\tilde{D}_s)\, dW_s = \alpha \int_0^T D_s\, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s\, dW_s$$

Proposition 6.21. $\displaystyle\int_0^{T_1} D_s\, dW_s + \int_{T_1}^{T_2} D_s\, dW_s = \int_0^{T_2} D_s\, dW_s$

Question 6.22. *If $D \geqslant 0$, then must $\int_0^T D_t\, dW_t \geqslant 0$?*

$\boxed{\text{NO}}$

Eg: $D_t = 1$ for all t.

then $\displaystyle\int_0^T 1\, dW_t = W_T - \cancel{W_0}^{\,0}$ could be ≤ 0.

Remark 6.23. (1) For Riemann-Stieltjes integrals $\frac{d}{dt}\left(\int_0^t D_r\, dS_r\right) = D_t S_t$. $D_t \frac{dS_t}{dt}$

(2) For Itô integrals: $\frac{d}{dt}\left(\int_0^t D_r\, dW_r\right)$ typically does not exist.

$$\rightarrow \frac{d}{dt}\left(\int_0^t D_s\, ds\right) \overset{\text{F.T.C.}}{=} D_t$$

$$\frac{d}{dt}\left(\int_0^t D_s\, dW_s\right) \text{ DNE } \left(\text{Eg: } D = 1 \quad \int_0^t 1\, dW = W_t \text{ not diff}\right)$$

6.5. Semi-martingales and Itô Processes.

Question 6.24. *What is* $\displaystyle\int_0^t W_s\, dW_s$*?*

IOU techniques to compute

Definition 6.25. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation (aka bounded variation).
- ▷ M is a martingale.

Definition 6.26. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s\,ds$, with $\int_0^t |b_s|\,ds < \infty$ ← (Riemann int)
- ▷ $M_t = \int_0^t \sigma_s\,dW_s$, with $\int_0^t |\sigma_s|^2\,ds < \infty$ ← Itô int.

Remark 6.27. Short hand notation for Itô processes: $\boxed{dX_t = b_t\,dt + \sigma_t\,dW_t.}$

Remark 6.28. Expressing $X = X_0 + B + M$ (or $dX = b\,dt + \sigma\,dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s.$$

$$\frac{d}{dt} X_t = \frac{d}{dt} X_0 + \frac{d}{dt}\int_0^t b_s\,ds + \frac{d}{dt}\int_0^t \sigma_s\,dW_s$$

$$\frac{dX_t}{dt} = 0 + b_t + \frac{d}{dt}\int_0^t \sigma_s \, dW_s$$

$$dX_t = b_t \, dt + \cancel{dt} \, \cancel{\frac{d}{dt}} \int_0^t \sigma_t \, dW_t$$

R-int

Ito int

$$\boxed{dX_t = b_t \, dt + \sigma_t \, dW_t}$$

Notational short hand for

$$\boxed{X_t - X_0 = \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s}$$

Theorem 6.29 (Itô formula). *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \overbrace{\partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t}^{\text{chain rule.}} + \overbrace{\frac{1}{2} \partial_x^2 f(t, X_t)\, d[X, X]_t}^{\text{QV.}}$$

Remark 6.30. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Stochastic Calc version of chain rule.

Itô correction.

Proposition 6.31. *If* $X = X_0 + B + M$, *then* $[X, X] = [M, M]$.

I.e. If $X_t - X_0 = \underbrace{\int_0^t b_s\, ds}_{\text{Ignore}} + \underbrace{\int_0^t \sigma_s\, dW_s}$

Then $\boxed{[X, X]_t = \int_0^t \sigma_s^2\, ds}$

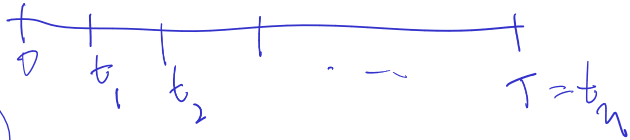
i.e. $\boxed{d[X, X]_t = \sigma_t^2\, dt = d[M, M]_t}$

Intuition $B \to$ Finite 1^{st} var: $V_{[0,T]} B$ is finite.

$$[X,X]_T = \lim_{\|P\|\to 0} \sum (\Delta_i X)^2 \qquad \Delta_i X = X_{t_{i+1}} - X_{t_i}$$

$[M,M]_T$

$$= \lim_{\|P\|\to} \Big(\sum (\Delta_i M)^2 + (\Delta_i B)^2 + 2(\Delta_i M)(\Delta_i B) \Big)$$



Claim: $\lim_{\|P\|\to 0} \sum (\Delta_i B)^2 \leftarrow 0 \leftarrow 1$

$$\lim_{\|P\| \to 0} \sum (\Delta_i B)(\Delta_i M) \to 0 \leftarrow ②$$

Check ①:

$$\sum (\Delta_i B)^2 \le \underbrace{\max_i (\Delta_i B)}_{\xrightarrow{\|P\| \to 0}\ 0 \ (\because B \text{ is cts})} \underbrace{\left(\sum_i |\Delta_i B| \right)}_{\xrightarrow{\|P\| \to 0}\ V_{[0,T]} B \ (\text{finite})}$$

$$\lim_{\|P\| \to 0}$$

$$\left(\Rightarrow [B, B]_T = 0 \right)$$

Check 2 $\lim\limits_{\|P\| \to 0} \sum (\Delta_i B)(\Delta_i M) = 0$

Cauchy-Schwarz

$$\left| \sum (\Delta_i B)(\Delta_i M) \right| \leq \underbrace{\left(\sum (\Delta_i B)^2 \right)^{1/2}}_{\xrightarrow{\|P\| \to 0}\ 0 = [B,B]_T} \underbrace{\left(\sum (\Delta_i M)^2 \right)^{1/2}}_{\xrightarrow{\|P\| \to 0}\ [M,M]_T}$$

$\Rightarrow$ 2.

Proposition 6.32 (Uniqueness). *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$, *with:*

- ▷ B, C *bounded variation*, $B_0 = C_0 = 0$
- ▷ M, N *martingale*, $M_0 = N_0 = 0$.

Then $X_0 = Y_0$, $B = C$ *and* $M = N$.

Use for us:

If $dX = b\,dt + \sigma\,dW_t$

& $dX = \tilde{b}\,dt + \tilde{\sigma}\,dW_t$

Then $b = \tilde{b}$

& $\sigma = \tilde{\sigma}$

Intuition :

$$\left.\begin{aligned} X &= X_0 + B + M \\ &= Y_0 + C + N \end{aligned}\right\} \quad B_0 = M_0 = C_0 = N_0 = 0 \quad \Rightarrow X_0 = Y_0$$

$$\Rightarrow B + M = C + N$$

$$\Rightarrow \underbrace{M - N}_{\text{Mg}} = \underbrace{C - B}_{\text{Finite 1}^{st}\text{ var.}} \quad (\Rightarrow 0 \text{ QV})$$

$\Rightarrow M - N$ is a mg & QV of $M - N = 0$ ($\&$ $M_0 - N_0 = 0$)

$$\Rightarrow \quad (M-N)^2 - \underbrace{[M-N, M-N]}_{0} \quad \text{is a mg}$$

$$\Rightarrow (M-N)^2 \quad \text{is a} \quad \text{mg}$$

$$\Rightarrow \quad E\underbrace{(M_t - N_t)}^2 = E(M_0 - N_0)^2 = 0$$

$$M_t = N_t \quad \text{a.s.} \quad \overset{||}{0} \;\; \overset{||}{0} \qquad (\Rightarrow B = C).$$

Corollary 6.33. *Let* $dX_t = b_t\,dt + \sigma_t\,dW_t$ *with* $\boldsymbol{E}\int_0^t b_s\,ds < \infty$ *and* $\boldsymbol{E}\int_0^t \sigma_s^2\,ds < \infty$*. Then* X *is a martingale if and only if* $b = 0$*.*

(Assume ...)

$dX = b\,dt + \sigma\,dW$ is mg $\iff$ $b = 0$

Check: Say X is a mg.

Then $X_t = X_0 + \underbrace{\int_0^t b_s\,ds}_{\text{fin } 1^{\text{st}} \text{ var}} + \underbrace{\int_0^t \sigma_s\,dW_s}_{\text{Mg}}$ $\Big\}$ Uniq $\Rightarrow$ $b = 0$!!

$= X_0 + 0 + (X_t - X_0)$

Definition 6.34. If $dX = b\,dt + \sigma\,dW$, define $\displaystyle\int_0^T D_t\,dX_t = \int_0^T D_t b_t\,dt + \int_0^T D_t \sigma_t\,dW_t$.

Remark 6.35. Note $\displaystyle\int_0^T D_t b_t\,dt$ is a *Riemann integral*, and $\displaystyle\int_0^T D_t \sigma_t\,dW_t$ is a *Itô integral*.

6.6. Itô's formula.

Remark 6.36. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$$

$$\frac{df(t,X_t)}{\cancel{dt}} = \partial_t f(t,X_t) \cdot \frac{dt}{\cancel{dt}} + \partial_x f(t,X_t)\, \frac{dX_t}{\cancel{dt}}$$

$f \in C^{1,2}$ means: $\partial_t f$ exists & is cts

$\partial_x f$ & $\partial_x^2 f$ exist & are cts.

Same terms from Chain Rule.

Theorem (Itô's formula, Theorem 6.29). *If $\boxed{f \in C^{1,2}}$, then*

$$df(t,X_t) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

Remark 6.37. If $dX_t = b_t\,dt + \sigma_t\,dW_t$ then

$$df(t,X_t) = \left(\partial_t f(t,X_t) + \cancel{b_t} + \frac{1}{2}\sigma_t^2 \cancel{}\right)dt + \partial_x f(t,X_t)\sigma_t\,dW_t\,.$$

$\partial_x f$ $\partial_x^2 f$

$$d f(t,X_t) = \partial_t f \, dt + \partial_x f \, dX_t + \frac{1}{2}\partial_x^2 f \, \underbrace{d[X,X]}_{\sigma_t^2\,dt}$$

$dX_t = b\,dt + \sigma\,dW$

Example 6.38. $\boxed{\text{Find the quadratic variation of } W_t^2.}$

Choose $f(t,x) = x^2$

$\partial_t f = 0$

$\partial_x f = 2x$

$\partial_x^2 f = 2$

$$d f(t, W_t) = d(W_t^2)$$

$$= \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, d[W,W]$$

$$\Rightarrow d(W_t^2) = 2W_t \, dW_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$\Rightarrow \boxed{d(W_t^2) = 2W_t\, dW_t + dt}$$

$$\Rightarrow d[W^2, W^2]_t = 4W_t^2\, dt ,$$

Recall: Semi mg: $X_t = X_0 + \underbrace{B_t}_{\text{Finite 1st var}} + \underbrace{M_t}_{\text{mg}}$ (cts adapted)

$$B_t = \underbrace{\int_0^t b_s \, ds}_{\text{R int}} \quad , \quad M_t = \underbrace{\int_0^t \sigma_s \, dW_s}_{\text{Ito int.}}$$

Notation: $dX_t = b_t \, dt + \sigma_t \, dW_t$

Notation: $\int_0^t D_s \, dX_s = \int_0^t D_s b_s \, ds + \int_0^t D_s \sigma_s \, dW_s.$

Theorem (Itô's formula, Theorem 6.29)**.** *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \boxed{\frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t}$$

Remark 6.37. If $dX_t = b_t\, dt + \sigma_t\, dW_t$ then

$$df(t, X_t) = \Big(\partial_t f(t, X_t) + b_t \partial_x f(t, X_t) + \frac{1}{2}\sigma_t^2 \partial_x^2 f(t, X_t)\Big)\, dt + \partial_x f(t, X_t)\sigma_t\, dW_t\,.$$

$f \in C^{1,2}$ means : $f = f(t, x)$

$\partial_t f$ exists & is cts

$\partial_x f$ & $\partial_x^2 f$ exist & are cts

$dX_t = b_t\, dt + \sigma_t\, dW$

$\Rightarrow d[X,X]_t = \sigma_t^2\, dt$

Intuition behind Itô's formula.

Itô $\iff$ $$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\,dt + \int_0^T \partial_x f(t, X_t)\,dX_t + \frac{1}{2}\int_0^T \partial_x^2 f(t, X_t)\,d[X, X]_t.$$

Consider a simple case: $f(t, x) = f(x)$

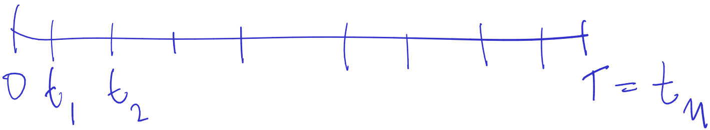
$$X_t = W_t.$$

Itô $\iff$ $$f(W_T) - f(W_0) = \int_0^T \partial_x f(W_t)\,dW_t + \frac{1}{2}\int_0^T \partial_x^2 f(W_t)\,dt.$$

Intuition behind Itô's formula.

Notation: Write $\partial_x f = f'$

Taylor's theorem: $f(x+h) - f(x) = h\, f'(x) + \frac{h^2}{2} f''(x) \boxed{+ O(h^3)}$



$$f(W_T) - f(W_0) = \sum_{0}^{n-1} f(W_{t_{k+1}}) - f(W_{t_k})$$

Taylor

$$\sum_{k=0}^{n-1} f'(W_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2} f''(W_{t_k})(W_{t_{k+1}} - W_{t_k})^2 + \text{small}.$$

$$\boxed{\Delta_k W = W_{t_{k+1}} - W_{t_k}.} = \underbrace{\sum_{k=0}^{n-1} f'(W_{t_k})\, \Delta_k W}_{\downarrow\, \|P\| \to 0} + \underbrace{\frac{1}{2} \sum_{k=0}^{n-1} f''(W_{t_k}) (\Delta_k W)^2}_{} + \text{small}$$

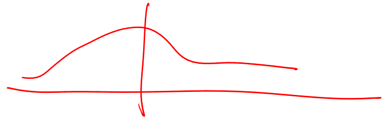
$$\int_0^T f'(W_t)\, dW_t$$

$$\approx \underbrace{\frac{1}{2} \sum_{k=0}^{n-1} f''(W_{t_k})(t_{k+1} - t_k)}_{\downarrow\, \|P\| \to 0}$$

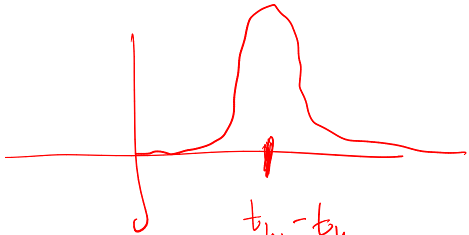
$$\frac{1}{2} \int_0^T f''(W_t)\, dt$$

$\Rightarrow$ Ito !!

Know $\quad \Delta_k W \sim N(0, t_{k+1} - t_k)$



$$(\Delta_k W)^2 \sim N(0, \underbrace{t_{k+1} - t_k}^{2})$$



$$\operatorname{Var}(\Delta_k W) = 2\underbrace{(t_{k+1} - t_k)^2}_{\longrightarrow 0}$$

Example 6.38. Find the quadratic variation of W_t^2.

Last time

Example 6.39. Find $\boxed{\int_0^t W_s \, dW_s}$.

Method: Guess a fn $f = f(t,x)$

so that $d\,\underline{f(t, W_t)} = \underline{\quad\quad}\, dt + \boxed{W_t \, dW_t}$

& integrate.

Note $d\,f(t, W_t) = \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2} \partial_x^2 f \, dt$

Choose $f(t,x) = \dfrac{x^2}{2}$.

$$\partial_t f = 0$$
$$\partial_x f = x$$
$$\partial_x^2 f = 1$$

$$d\left(\frac{W_t^2}{2}\right) = d f(t, W_t)$$
$$= \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2} \partial_x^2 f \, d[W, W]_t$$
$$= 0 + W_t \, dW_t + \frac{1}{2} dt$$

$$\Rightarrow \frac{W_T^2}{2} - 0 = \int_0^T W_t \, dW_t + \frac{T}{2}.$$

$$\boxed{\Rightarrow \int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}}$$

Example 6.40. Let $M_t = W_t$, and $N_t = W_t^2 - t$.

- ▷ We know M, N are martingales.
- ▷ Is MN a martingale?

Opt ① $\longrightarrow$ $M_t N_t = W_t^3 - t W_t$ & Compute $E_s(\quad)$

$$E_s(W_t^3) = E_s\big((W_t - W_s + W_s)^3\big)$$ & expand & suffer.

Better way: Use Ito:

Compute $d(M_t N_t) = d(W_t^3 - t W_t)$

Choose $f(t,x) = x^3 - tx$

$$\partial_t f = -x$$
$$\partial_x f = 3x^2 - t$$
$$\partial_x^2 f = 6x$$

Ito: $d(W_t^3 - tW_t) = \partial_t f\, dt + \underline{\partial_x f}\, dW + \frac{1}{2}\partial_x^2 f\, d[W,W]$

$$= -W_t\, \underline{dt} + (3W_t^2 - t)\, dW + \frac{1}{2}\, 6W_t\, \underline{dt}$$

$$= \underbrace{(3W_t - W_t)}\, \underline{dt} + (3W_t^2 - t)\, dW_t$$

Coeff of $dt \neq 0 \Rightarrow$ MN can NOT be a mg.

Example 6.41. Let $X_t = t\sin(W_t)$. Is $X_t^2 - [X, X]_t$ a martingale?

Example 6.42. Say $dM_t = \sigma_t dW_t$. Show that $\underbrace{M^2 - [M, M]}$ is a martingale.

$(M \to mg)$ Check: $X_t = M_t^2 - \underbrace{[M, M]_t}$,

$f(t, x) = x^2 - \underbrace{[M, M]_t}$

(Recall: $[M, M]_t = \int_0^t \sigma_s^2 \, ds$

$\Leftrightarrow \boxed{d[M, M] = \sigma_t^2 \, dt}$ (diff))

Apply Ito: $d f(t, M) = \partial_t f dt + \partial_x f \, dM + \boxed{\frac{1}{2} \partial_x^2 f \, d[M, M]}.$

$$= \cancel{-\sigma_t^2\, dt} + 2M_t\, dM_t + \boxed{\tfrac{1}{2}\cdot 2\, \cancel{\sigma_t^2\, dt}}$$

$$= 2M_t\, \sigma_t\, dW_t$$

$\partial_t f = -\sigma_t^2$

$\boxed{\partial_x f = 2x}$

$\partial_x^2 f = 2.$

No dt term → is a mg!

Theorem 6.43 (Lévy's criterion)**.** *Let M be a continuous martingale such that $\boxed{M_0 = 0}$ and $\boxed{[M, M]_t = t}$. Then M is a Brownian motion.*

Know W →

1. is cts
2. is a mg
3. $d[W,W]_t = dt$

I.e. $M_t - M_s \sim N(0, t-s)$

& $M_t - M_s$ is ind of $\mathcal{F}_s$.

① Show $M_t \sim N(0,t)$

Will show $MGF(M_t) = MGF(N(0,t))$

$$\varphi_\lambda(t) = E\, e^{\lambda M_t}$$

X is a RV.

Then $\varphi = MGF$ of X

$$\varphi_\lambda(\cancel{t}) = E\, e^{\lambda X}$$

$$\varphi_\lambda(N(0,t)) = e^{\lambda^2 t/2}$$

Apply Itô to $d\left(e^{\lambda M_t}\right)$:

$$\Rightarrow d\left(e^{\lambda M_t}\right) = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2}\partial_x^2 f \, d[M,M]$$

Choose $f(t,x) = e^{\lambda x}$

$\partial_t f = 0$

$\partial_x f = \lambda e^{\lambda x}$

$\partial_x^2 f = \lambda^2 e^{\lambda x}$

$$= 0 + \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} e^{\lambda M_t} dt$$

(Given $d[M,M] = dt$)

$$\Rightarrow \mathbb{E} e^{\lambda M_T} - 1 = \mathbb{E}\int_0^T \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2}\mathbb{E}\int_0^T e^{\lambda M_t} dt$$

$$\varphi_\lambda(t) = E e^{\lambda M_t}$$

Say $dM = \sigma\, dW_t$

$$\Rightarrow E\int_0^T \lambda e^{\lambda M_t}\, dM_t = E\int_0^T \lambda e^{\lambda M_t} \sigma_t\, dW_t$$

$$= 0 \quad (\because \text{Itô int is a mg})$$

$$\Rightarrow \varphi_\lambda(T) - 1 = 0 + \frac{\lambda^2}{2} E\int_0^T e^{\lambda M_t}\, dt$$

$$= \frac{\lambda^2}{2} \int_0^T \underbrace{E e^{\lambda M_t}}_{\varphi_\lambda(t)} dt$$

$$\varphi_\lambda(T) - 1 = \frac{\lambda^2}{2} \int_0^T \varphi_\lambda(t)\, dt$$

diff wrt T : $\quad \varphi_\lambda'(T) = \frac{\lambda^2}{2} \varphi_\lambda(T)$

$$\Rightarrow \varphi_\lambda(T) = \varphi_\lambda(0) \cdot \exp\left(\frac{\lambda^2}{2} T\right)$$

$$\Rightarrow \varphi_\lambda(T) = e^{\frac{\lambda^2}{2}T} = \text{MGF of normal!!}$$

$$\Rightarrow M_t \sim N(0,t)$$

Try same strategy & show $M_t - M_s \sim N(0, t-s)$

& $M_t - M_s$ is ind of $\mathcal{F}_s$.

Let $\varphi(t) = E_s\, e^{\lambda(M_t - M_s)}$

Compute φ: (Ito $\Rightarrow$)

$$d\left(e^{\lambda M_t}\right) = \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} e^{\lambda M_t} dt$$

Int from s to t:

$$e^{\lambda M_t} - \underline{\underline{e^{\lambda M_s}}} = \lambda \int_s^t e^{\lambda M_r} dM_r + \frac{\lambda^2}{2} \int_s^t e^{\lambda M_r} dr$$

$$\Rightarrow E_s e^{\lambda(M_t - M_s)} - 1 = \lambda E_s \int_s^t e^{\lambda(M_r - M_s)} \underline{\underline{dM_r}} + \frac{\lambda^2}{2} E_s \int_s^t e^{\lambda(M_r - M_s)} dr$$

adapted so OK.

$$\varphi(t) = E_s\, e^{\lambda(M_t - M_s)}$$

$$\Rightarrow \varphi(t) - 1 = \underbrace{\lambda \int_s^t (\quad)}_{0} + \frac{\lambda^2}{2} \int_s^t E_s\, e^{\lambda(M_r - M_s)}\, dr$$

$$\boxed{\Rightarrow \varphi(t) = 1 + \frac{\lambda^2}{2} \int_s^t \varphi(r)\, dr}$$ $\leftarrow$ same eq as before

$$\Rightarrow \varphi(t) = e^{\frac{\lambda^2}{2}(t-s)}$$ = MGF of normal.

$$\Rightarrow \quad E_s\, e^{\lambda(M_t - M_s)} = e^{\frac{\lambda^2}{2}(t-s)}$$

$$\Rightarrow \quad E\, e^{\lambda(M_t - M_s)} = E\, E_s\, e^{\lambda(M_t - M_s)} = \underbrace{e^{\frac{\lambda^2}{2}(t-s)}}_{\text{MGF of } N(0,\,t-s).}$$

Indep:

X & Y are indep $\iff E e^{\lambda X + \mu Y} = E e^{\lambda X}\, E e^{\mu Y}$

If X is $\mathcal{F}_s$ meas

$$E\left(e^{\mu X + \lambda(M_t - M_s)}\right)$$

$$= E\, E_s\left(e^{\mu X}\, e^{\lambda(M_t - M_s)}\right)$$

$$= \underbrace{E\, e^{\mu X}}_{\text{MGF}(X)} \left(E_s\, e^{\lambda(M_t - M_s)} \right) \to e^{\frac{\lambda^2}{2}(t-s)}$$

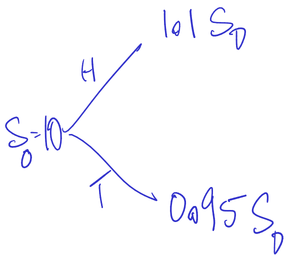
$\Rightarrow$ Indep !.

7. Review Problems

Problem 7.1 (From 2021 Midterm). Consider a discrete time market consisting of a bank and a stock. The bank pays interest rate $r = 5\%$ at every time period. Let S_n denote the stock price at time n, and we know $S_0 = \$10$. The stock price changes according to the flip of a fair coin: if the coin lands heads the stock price increases by 10% (i.e. $S_{n+1} = 1.1S_n$), and if the coin lands tails the stock price decreases by 5% (i.e. $S_{n+1} = 0.95S_n$). An option pays the holder $\boxed{S_N^3}$ at time $N = 5$. Find the arbitrage free price of this option at time $n = 1$. Also find the number of shares held in the replicating portfolio at time $n = 0$. Round your final answer two decimal places. (I recommend rounding intermediate steps to three decimal places.)

Bank $\to$ $r = 5\%$

Stock $\to$ $S_0 = \$10$



$u = 1.1$

$d = 0.95$

$r = 0.05$

Payoff $V_N = S_N^3$ $(N = 5)$

Find V_1 = AFP at $n = 1$

Formula: V_n = AFP at time n $= \underbrace{\frac{1}{D_n}\tilde{E}_n(D_N V_N)}$

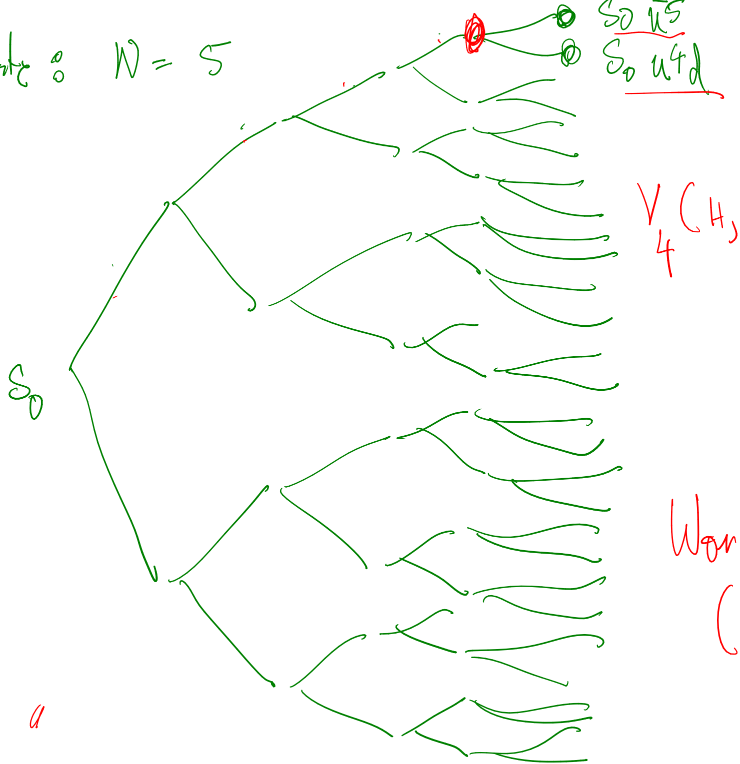
$$D_n = (1+r)^{-n}$$

$\tilde{E} \rightarrow$ coin flips with prob. $\tilde{p}$ heads, $\tilde{q}$ tails

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1+r}{u-d}$$

n

Compute: $N = 5$



$$V_4(H,H,H,H,*) = \frac{1}{1+r}\tilde{E}_4(S_5)$$

$$= \frac{1}{1+r}\left(\tilde{p}\, S_0 u^5 + \tilde{q}\, S_0 u^4 d\right)$$

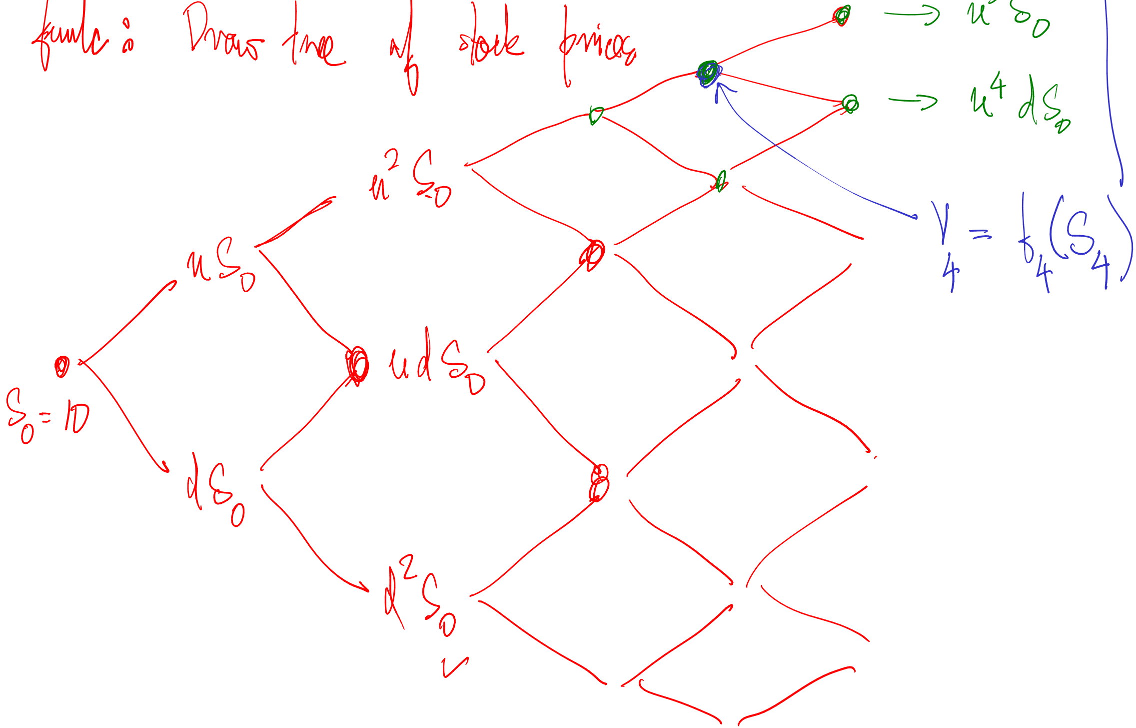
Works, but too much "work"
(Without a computer)

Better Strategy: From HW.

Know if $V_N = f_N(S_N)$

then $V_n = f_n(S_n)$ & $\boxed{f_n(x) = \dfrac{f_{n+1}(ux)\tilde{p} + f_{n+1}(xd)\tilde{q}}{1+r}}$

Use funct.: Draw tree of stock prices



$S^3_5 = V_5$

$S^3_5 = V_5$

$$f_4(x) = \frac{f_5(ux)\tilde{p} + f_5(dx)\tilde{q}}{1+r} = \frac{(ux)^3\tilde{p} + (dx)^3\tilde{q}}{1+r}$$

$(x = S_4)$ & compute.

Even shorter method:

Know $\quad V_n = \frac{1}{D_n}\tilde{E}_n(D_N V_N)$

Used $D_n V_n$ is a mg under $\tilde{P}$

Know $D_n V_n = \tilde{E}_n (D_{n+1} V_{n+1})$

$$D_n = (1+r)^{-n} \Rightarrow V_n = \frac{1}{1+r} \tilde{E}_n (\cancel{D_{n+1}} V_{n+1})$$

$$V_{N-1} = \frac{1}{1+r} \tilde{E}_{N-1} (V_N) = \underbrace{\frac{1}{1+r} \tilde{E}_{N-1} (S_N^3)}$$

Let $X_n = \begin{cases} u & \text{if } n^{th} \text{ coin is heads} \\ d & \text{if } n^{th} \text{ coin is tails} \end{cases}$

Then $S_{n+1} = \underbrace{S_n}_{\mathcal{F}_n\text{-meas}} \underbrace{X_{n+1}}_{\text{indep of } \mathcal{F}_n.}$

So $\frac{1}{1+r}\tilde{E}_{N-1}\left(S_N^3\right) = \frac{1}{1+r}\tilde{E}_{N-1}\left(S_{N-1}^3 X_N^3\right)$

$$= \frac{1}{1+r} S_{N-1}^3 \, \tilde{E}_{\cancel{N-1}} X_N^3$$

($\because S_{N-1}$ is $\mathcal{F}_{N-1}$ meas, X_N is indep)

$$\boxed{\Rightarrow \quad V_{N-1} = \underline{S_{N-1}^3} \underbrace{\left(\frac{u^3\tilde{p} + d^3\tilde{q}}{1+r}\right)}}$$

Repeat: $V_{N-2} = \frac{1}{1+r}\tilde{E}_{N-2} V_{N-1} = \frac{u^3\tilde{p} + d^3\tilde{q}}{(1+r)^2} \underbrace{\tilde{E}_{N-2} S_{N-1}^3}$

$$= \frac{(u^3\tilde{p} + d^3\tilde{q})}{(1+r)^2} S^3_{N-2} (u^3\tilde{p} + d^3\tilde{q})$$

$$= \left(\frac{u^3\tilde{p} + d^3\tilde{q}}{1+r}\right)^2 S^3_{N-2}$$

$$\therefore V_1 = V_{N-4} = \left(\frac{u^3\tilde{p} + d^3\tilde{q}}{1+r}\right)^4 S^3_1$$

Problem 7.2. If $0 \leqslant r \leqslant \underline{s \leqslant t}$, find $\boldsymbol{E}(W_s W_t)$ and $\boxed{\boldsymbol{E}(W_r W_s W_t)}$.

Compute $E(W_s W_t) = s \wedge t = s$

Check: $E(W_s W_t) = E\big(\underbrace{W_s (W_t - W_s)}_{\text{indep}} + W_s^2\big)$

$= \underline{EW_s}\, E(W_t - W_s) + EW_s^2$

$= 0 + s \qquad (W_s \sim N(0,s))$

Check 2: $E(W_s W_t) = E\, E_s(W_s W_t) \quad$ (tower)

$$= E\Big(W_s \underbrace{E_s W_t}_{mg}\Big) = E(W_s \cdot W_s) = s.$$

Compute $E(W_r W_s W_t)$

$$= E\big(E_r\, W_r W_s W_t\big)$$

$$= E\Big(W_r\, E_r\big(W_s W_t\big)\Big)$$

$$= E\Big(W_r\, E_r\, E_s\big(W_s W_t\big)\Big)$$

$$= E\Big(W_r\, E_r\big(W_s\, E_s W_t\big)\Big)$$

$$= E\left(W_r\, E_r\left(W_{\underline{s}}^2\right)\right)$$

$$= E\left(W_r\, E_r\left(\underbrace{W_s^2 - s}_{mg} + \underline{s}\right)\right)$$

$$= E\left(W_r\left(W_r^2 - r + s\right)\right)$$

$$= E\,\underline{W_r^3} + E\,W_r(s - r) = 0.$$

Problem 7.3. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2}\,ds\,, \quad \boxed{Y_t = \exp\Big(\int_0^t W_s\,ds\Big)\,,} \quad Z_t = tX_t^2$$

Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

Itô decompose X:

Write $X_t = f(t, W_t)$ where $\boxed{f(t,x) = \int_0^x e^{-s^2}\,ds}$

Need $\partial_t f$, $\partial_x f$, $\partial_x^2 f$ to exist

$\partial_t f = 0$

$\partial_x f = e^{-x^2}$ $\quad$ (FTC: $\partial_x \int_0^x g(s)\,ds = g(x)$)

$$\partial_x^2 f = e^{-x^2}(-2x)$$

Itô: $dX = d f(t, W_t) = \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2} \partial_x^2 f \, d[W, W]$

$$= 0 + e^{-W_t^2} dW_t + \frac{1}{2}\left(e^{-W_t^2}(-2W_t)\right) dt$$

$$dX_t = \underline{e^{-W_t^2} dW_t} - W_t e^{-W_t^2} dt$$

$$\Rightarrow X_t = X_0 + \int_0^t e^{-W_s^2} dW_s - \int_0^t W_s e^{-W_s^2} ds$$

Mg   Finite 1^{st} var.

Remember → Ito ints are Mg's

& Riemann int are finite 1^{st} var.

$$Y_t = \exp\left(\int_0^t W_s \, ds\right) \qquad \text{Find Ito decomp.}$$

$$Y_t = f(t, W_t) \quad \& \quad f(t,x) = \begin{cases} \exp\left(\int_0^t W_s \, ds\right) \\ \cancel{\exp\left(\int_0^t x \, ds\right)} \end{cases}$$

$$\partial_x f(t,x) = 0$$

$$\partial_x^2 f(t,x) = 0$$

$$\partial_t f(t,x) = \exp\left(\int_0^t W_s\,ds\right) \frac{d}{dt}\int_0^t W_s\,ds = W_t$$

$$= f(t,x) \quad W_t.$$

$$\therefore\ dY = \partial_t f\,dt + \underbrace{\partial_x f}_{0}\,dW_t + \frac{1}{2}\underbrace{\partial_x^2 f}_{0}\,d[W,W]$$

✓

$$= f(t, W_t) \cdot W_t \, dt = Y_t \, W_t \, dt$$

$$\Rightarrow Y_t = Y_0 + \underbrace{\int_0^t Y_s W_s \, ds}_{\text{Finite 1st var}} + \underbrace{0}_{\text{Mg.}}$$

Problem 7.4. Define the processes X, Y by

$$X_t \stackrel{\text{def}}{=} \int_0^t W_s \, ds \,, \quad \boxed{Y_t \stackrel{\text{def}}{=} \int_0^t W_s \, dW_s \,.}$$

Given $0 \leqslant s < t$, compute $\boldsymbol{E} X_t$, $\boldsymbol{E} Y_t$, $\boldsymbol{E}_s X_t$, $\boldsymbol{E}_s Y_t$.

$$EX_t = E\int_0^t W_s\,ds = \int_0^t (EW_s)\,ds = 0$$

$$E_s X_t = E_s \int_0^t W_r\,dr = \int_0^t E_s W_r\,dr$$

$$= \int_0^s E_s W_r\,dr + \int_s^t E_s W_r\,dr$$

Rules:

$$\Rightarrow E\int_0^t b_s\,ds = \int_0^t E_s b_s\,ds$$

$$E_s \int_0^t b_r\,dr = \int_0^t E_s b_r\,dr$$

$$E_s \int_0^t \sigma_r\,dW_r = \int_0^s \sigma_r\,dW_r$$

Ito int are mg's.

$$= \int_0^s W_r \, dr + \int_s^t W_s \, dr$$

$$= \int_0^s W_r \, dr + W_s (t - s)$$

$$E \int_0^t W_r \, dW_r = E \int_0^0 W_r \, dW_r = 0 \qquad (Mg)$$

$$E_s \int_0^t W_r \, dW_r = \int_0^s W_r \, dW_r \qquad (Mg)$$

Problem 7.5. Let $M_t = \int_0^t W_s \, dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\Big(M_t - \int_0^t f(s, W_s) \, ds\Big)$$

is a martingale.

Problem 7.6. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 \, dt$.

$$X_t = \int_0^t \sigma_u \, dW_u \, .$$

(1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leqslant s < t$ compute $\boldsymbol{E} e^{\lambda M_t}$ and $\boldsymbol{E}_s e^{\lambda M_t - M_s}$
(2) If $r \leqslant s$ compute $\boldsymbol{E} \exp(\lambda M_r + \mu(M_t - M_s))$.
(3) What is the joint distribution of $(M_r, M_t - M_s)$?
(4) *(Lévy's criterion)* If $d[M, M]_t = dt$, then show that M is a standard Brownian motion.

Problem 7.7. Define the process X, Y by

$$X = \int_0^t s\,dW_s\,, \quad Y = \int_0^t W_s\,ds\,.$$

Find a formula for $\boldsymbol{E}X_t^n$ and $\boldsymbol{E}Y_t^n$ for any $n \in \mathbb{N}$.

$\boxed{\text{Claim: Both } X_t \text{ \& } Y_t \text{ are normal!}}$

Can find EX_t^n & EY_t^n if I know EX_t & EX_t^2 & EY_t & EY_t^2.

Find EX_t & EX_t^2:

a

$$\mathbb{E}X_t = \mathbb{E}\int_0^t s\,dW_s = 0$$

$$\mathbb{E}X_t^2 = \mathbb{E}\left(\int_0^t \underline{\underline{s}}\,d\underline{\underline{W_s}}\right)^2 = \mathbb{E}\int_0^t s^2\,ds = \frac{t^3}{3}$$

for Y:

$$\mathbb{E}Y_t = \mathbb{E}\left(\int_0^t W_s\,ds\right) = \int_0^t \mathbb{E}W_s\,ds = 0$$

$$\mathbb{E}Y_t^2 = \mathbb{E}\left(\int_0^t W_s\,ds\right)^2$$

$$\Rightarrow E\left[\left(\int_0^t W_s\, ds\right)\left(\int_0^t W_s\, ds\right)\right]$$

$$= E\left[\left(\int_0^t W_s\, ds\right)\left(\int_0^t W_r\, dr\right)\right]$$

$$= E\int_{s=0}^{t}\int_{r=0}^{t} W_s W_r \, ds\, dr = \int_{s=0}^{t}\int_{r=0}^{t} E(W_s W_r)\, ds\, dr$$

4

$$= \int_{s=0}^{t} \int_{r=0}^{t} (s \wedge r)\, ds\, dr$$ & compute.

Why are X & Y normal:

Not random

Normal

$$X_t = \int_0^t s\, dW_s = \lim \sum t_k (W_{t_{k+1}} - W_{t_k}) \longrightarrow \text{Normal.}$$

Sum of ind normals.

2

$$Y_t = \int_0^t W_s \, ds = \underbrace{\lim_{\|P\| \to 0} \underbrace{\sum \underbrace{W_{t_k}}_{\text{normal}} (t_{k+1} - t_k)}_{\text{sum of normals}}}_{\text{Normal.}}$$

u

$$\mathbb{E}_s \int_0^t \sigma_r \, dW_r = \cancel{\int_0^t \mathbb{E}_s\left(\sigma_r \, dW_r\right)}$$

may not make sense!

Problem 7.8. Let $M_t = \int_0^t W_s \, dW_s$. For $s < t$, is $M_t - M_s$ independent of $\mathcal{F}_s$? Justify.

Problem 7.9. Determine whether the following identities are true or false, and justify your answer.

(1) $e^{2t}\sin(2W_t) = 2\int_0^t e^{2s}\cos(2W_s)\,dW_s$.

(2) $|W_t| = \int_0^t \operatorname{sign}(W_s)\,dW_s$. (Recall $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 0$ if $x = 0$.)

Midterm: Went well

8. Black Scholes Merton equation

8.1. Market setup and assumptions.

- Cash: simple interest rate r in a bank.
- Let Δt be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$.
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t) C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\partial_t C = rC$ and $\boxed{C_t = C_0 e^{rt}}$.
- r is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $\boxed{r = \frac{1}{T} \ln(1+\rho)}$.

$$\boxed{t = n\Delta t}$$

$$C_{(n+1)\Delta t} = (1 + r\Delta t)\, C_{n\Delta t}$$

$$\iff \frac{C_{(n+1)\Delta t} - C_{n\Delta t}}{\Delta t} = r\, C_{n\Delta t}$$

$$\Delta t \to 0: \quad \partial_t C_t \qquad\qquad C_t.$$

$$(1+\rho) = e^{rT}$$

$$\iff r = \frac{1}{T} \ln(1+\rho)$$

- Stock price: $S_{t+\Delta t} = (1 + \alpha\Delta t)S_t + \text{noise}.$
  - ▷ Variance of noise should be proportional to Δt.
  - ▷ Variance of noise should be proportional to S_t.
- $S_{t+\Delta t} - S_t = \alpha S_t\,\Delta t + \sigma S_t(\Delta W_t).$

$$S_{t+\Delta t} = (1 + \alpha\Delta t)S_t + \underbrace{\sigma S_t \boxed{\Delta W_t}}_{\text{noise}}.$$

GBM

Definition 8.1. A Geometric Brownian motion with parameters α, σ is defined by:

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t\,.$$

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Typical first model for stock price.

Proposition 8.2. $\boxed{S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$

Compute: $dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$

$$\Rightarrow \frac{dS_t}{S_t} = \alpha \, dt + \sigma \, dW_t$$

$f(x) = \ln x$

$\partial_t f = 0$

$\partial_x f = \frac{1}{x}$

$\partial_x^2 f = -\frac{1}{x^2}$

Apply Itô to $\ln(S_t)$:

$$d(\ln S_t) = 0 + \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} \underbrace{d[S,S]_t}_{\sigma^2 S_t^2 \, dt}$$

u

$$\Rightarrow \quad d(\ln S_t) = \frac{1}{S_t}\left(\alpha S_t\, dt + \sigma S_t\, dW_t\right) - \frac{1}{2S_t^2}\sigma^2 S_t^2\, dt$$

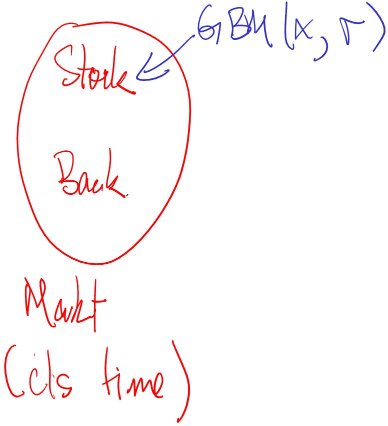
$$= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

Integrate $\Rightarrow \underbrace{\ln S_t - \ln S_0}_{\ln\left(\frac{S_t}{S_0}\right)} = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t$

$$\boxed{\Rightarrow S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)}$$

Market Assumptions.

- 1 stock, Price S_t, modelled by GBM(α, σ).
- Money market: Continuously compounded interest rate $\boxed{r}$.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = rC_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)



8.2. **The Black, Sholes, Merton equation.** Consider a security that pays $\boxed{V_T = g(S_T)}$ at maturity time T.

Theorem 8.3. *If the security can be replicated, and $f = f(t, x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{8.1}$$

(B.S. PDE).

$$f(t, 0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{8.2}$$

(Boundary cond).

$$\boxed{f(T, x) = g(x)} \qquad x \geqslant 0\,. \tag{8.3}$$

Theorem 8.4. *Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 8.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 8.6. $g(x) = (x - K)^+$ is a European call with strike K and maturity T.

Remark 8.7. $g(x) = (K - x)^+$ is a European put with strike K and maturity T.

At maturity

$X_T = f(T, \cancel{S_T})$ x

$= g(\cancel{S_T})$ x

(8.2): If $S_t = 0$ then for all later times $S_t = 0$

$\Rightarrow S_T = 0 \Rightarrow$ Payoff $V_T = g(0)$

AFP at time $t = g(0)\, e^{-r(T-t)}$ (discounting). $\rightarrow$ 8.2

Proposition 8.8. *A standard change of variables gives an explicit solution to* (8.1)–(8.3)*:*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}\,, \qquad \boxed{\tau = T - t\,.} \tag{8.4}$$

Corollary 8.9. *For European calls,* $g(x) = (x-K)^+$*, and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

is the CDF of a standard normal variable.

Will derive (8.4) & (8.5) using RNM in last lecture.

Remark 8.10. Equation (8.1) is called a *partial differential equation.* In order to have a unique solution it needs:

(1) A terminal condition (this is equation (8.3)),
(2) A boundary condition at $x = 0$ (this is equation (8.2)),
(3) A boundary condition at infinity (not discussed yet).

- ▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is

$$\lim_{x \to \infty} f(t, x) = 0 .$$

- ▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is

$$\lim_{x \to \infty} \left[f(t, x) - (x - Ke^{-r(T-t)}) \right] = 0 \qquad \text{or} \qquad f(t, x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty .$$

Definition 8.11. If X_t is the wealth of a self-financing portfolio then

$$\boxed{dX_t = \Delta_t\, dS_t + r(X_t - \Delta_t S_t)\, dt}$$

for some adapted process Δ_t (called the trading strategy).

Disc time: Self fin means

$$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r)$$

$$\iff X_{n+1} - X_n = \Delta_n (S_{n+1} - S_n) + r(X_n - \Delta_n S_n)$$

$$\boxed{dX_t = \Delta_t\, dS_t + r(X - \Delta_t S_t)\, \boxed{dt}}$$

Proof of Theorem 8.3. If $X_t = f(t, S_t)$ = wealth of R. Port

then f satisfies BS PDE 8.1 8.2 8.3

$$\boxed{\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f}$$

$$\left\{ \begin{aligned} dS_t &= \alpha S_t\, dt + \sigma S_t\, dW_t \\ d[S,S] &= \sigma^2 S_t^2\, dt \end{aligned} \right.$$

1 Know $X_t = f(t, S_t)$

Compute dX_t using Itô

$$d f(t, S_t) = \partial_t f\, dt + \partial_x f\, dS_t + \frac{1}{2} \partial_x^2 f\, d[S,S].$$

$$\Rightarrow d f(t, S_t) = \partial_t f \, dt + \partial_x f \left(\alpha S_t \, dt + \sigma S_t \, dW_t \right) + \frac{1}{2} \partial_x^2 f \, \sigma^2 S_t^2 \, dt$$

$$\boxed{d f(t, S_t) = \left(\partial_t f + \alpha \partial_x f \, S_t + \frac{\sigma^2}{2} \partial_x^2 f \, S_t^2 \right) dt + \left(\sigma \partial_x f \, S_t \right) dW_t} \quad (*)$$

(2) Know X is self fin

$$\Rightarrow dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t) \, dt$$

$$= \Delta_t \left(\alpha S_t \, dt + \sigma S_t \, dW_t \right) + r(X_t - \Delta_t S_t) \, dt$$

$$\boxed{\Rightarrow dX_t = \left(rX_t + (\alpha - r)\Delta_t S_t\right)dt + \sigma \Delta_t S_t \, dW_t} \quad (**)$$

$\boxed{\Delta_t = \text{\# shares at time } t \text{ in Rep port.}}$

(3) Equate $(*)$ & $(**)$

Uniq of Itô $\Rightarrow$ the dt terms must be equal

& the dW " " " "

Equate dW terms: $\Rightarrow \cancel{\sigma}\,\Delta_t \cancel{S_t} = \cancel{\sigma}\,\partial_x f \cancel{S_t}$

$$\boxed{\Rightarrow \Delta_t = \partial_x f(t, S_t)} \leftarrow \text{Delta Hedging Rule.}$$

Equate dt terms:

$$\Rightarrow rX_t + (\alpha - r)\Delta_t S_t = \partial_t f + \alpha S_t \partial_x f + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

Substitute $X_t = f(t, S_t)$ & $\Delta_t = \partial_x f$

$$\Rightarrow rf + (\cancel{\alpha} - r) S_t \partial_x f = \underline{\partial_t f} + \cancel{\alpha S_t \partial_x f} + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

$$\Rightarrow r f = \partial_t f + r S \partial_x f + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

Write x instead of S_t get B.S. PDE (8.1) !

Proof of Theorem 8.4. Assume f solves BS PDE

Want to show $f(t, S_t) = \text{AFP} = \text{Wealth of R. Port.}$

① Let $Y_t = f(t, S_t)$

② Let X_t = wealth of a self fin port

Choose $X_0 = f(0, S_0) = Y_0$

Choose Δ_t = # shares held at time t

$= \partial_x f(t, S_t)$ (Delta Hedging)

<u>Claim:</u> X is the Rep Port.

i.e $X_T = g(S_T) \overset{(8.3)}{=} f(T, S_T) = Y_T$.

Will show: For all $t \geq 0$, $X_t = Y_t = f(t, S_t)$.

① Compute dX_t?

$$dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt$$

$$= \Delta_t (\alpha S_t \, dt + \sigma S_t \, dW_t) + r(X - \Delta S)\,dt$$

$$\boxed{dX_t = \left((\alpha - r)\Delta_t S_t + r X_t\right) dt + \sigma \Delta_t S_t \, dW_t}$$

② Compute dY: $\quad Y = f(t, S_t)$

$$\Rightarrow dY = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2}\partial_x^2 f \, d[S,S]$$

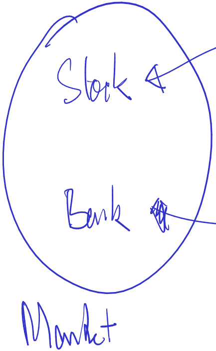
$$= \partial_t f \, dt + \partial_x f \left(\alpha S \, dt + \sigma S \, dW\right) + \frac{1}{2}\partial_x^2 f \, \sigma^2 S^2 \, dt$$

$$dY_t = \left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2}{2} S^2 \partial_x^2 f\right) dt + \sigma \partial_x f\, S\, dW$$

$$\Rightarrow d(Y - X) = \left(\qquad \right) dt + \overset{0}{\cancel{\left(\sigma \partial_x f\, S - \sigma \Delta_t S\right)}} dW$$

$$= \left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2} \partial_x^2 f - (\alpha - r) \Delta_t S_t - r X_t\right) dt + 0\, dW$$

Last time: Cts time market



GBM(α, σ): $dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$

Int rate r: $dC_t = r C_t$ $\left(\iff C_t = C_0 e^{rt} \right)$

B.S. PDE:

Security pays $V_T = g(S_T)$ at maturity T

Thm 1 If sec can be rep & $X_t = AFP = f(t, S_t)$

then f satisfies

1. $\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f$ PDE

2. $f(x, T) = g(x)$ (Terminal cond)

3. $f(0, t) = g(0) e^{-r(T-t)}$ (Boundary cond)

Thm 2: If f solves (1) (2) & (3)

then the sec can be rep

& X_t = wealth of R. Port = AFP = $f(t, S_t)$

Last time: Proved (1) $\longrightarrow$

X self fin if $dX_t = \Delta_t \, dS_t + (\cancel{\ } r)(X_t - \Delta_t S_t)\,dt$

Know $X_t = f(t, X_t)$ is self fin

Compute dX_t by self fin $\Big\}$ equate & get B.S. PDE.
& $d f(t, S_t)$ by Itô

Thm 2: Assume f solves the BS PDE (1, 2, 3)

Claim: $f(t, S_t)$ = wealth of the R. Port

Last time: let X_t = wealth of a self fin Port with

$$X_0 = f(0, S_0)$$

~~*Proof of Theorem 8.4 (without discounting).*~~

$$\& \quad \Delta_t = \partial_x f(t, S_t)$$

$$dX_t = \Delta_t \, dS + r(X_t - \Delta_t S_t) \, dt$$

$$= \underline{\qquad\qquad\qquad}$$

Let $Y_t = f(t, S_t)$ (Want to show $X_t = Y_t$)

If we know $X_t = Y_t$ then $X_T = Y_T = f(T, S_T) = g(S_T)$

$= V_T$

$\Rightarrow X_t = f(t, S_t)$ = wealth of R port = AFP

$\to$ Compute dY_t by Itô: $dY_t = ?\,dt + \ldots$

Then simplify & write $d(Y - X)$

$$d(Y_t - X_t) = \Big(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2}\partial_x^2 f$$

$$- (\alpha - r)\Delta_t S_t - r X_t\Big)dt + 0\, dW$$

$$\boxed{\text{Recall } \partial_x f(t, S_t) = \Delta_t}$$

$$= \Big(\underbrace{\partial_t f + r S_t \partial_x f + \frac{\sigma^2 S^2}{2}\partial_x^2 f}_{r f} - r X_t\Big)dt$$

$$\Rightarrow d(Y_t - X_t) = \big(r f(t, S_t) - r X_t\big)dt = r(Y_t - X_t)dt$$

$$\Rightarrow \quad Y_t - X_t = (Y_0 - X_0)e^{rt}$$

$$= 0 \quad \Rightarrow \text{done!}$$

$$\boxed{\begin{array}{l}\text{Remember by choice} \\ X_0 = f(0, S_0) = Y_0\end{array}}$$

Remark 8.12. The arbitrage free price does not depend on the mean return rate!

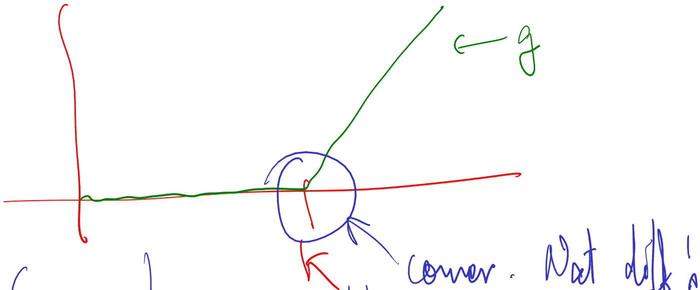
$$dS_t = \alpha S\,dt + \sigma S\,dW_t$$

mean ret rate

$$\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f \leftarrow \text{no } \alpha!$$

Question 8.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (8.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

$g(x) = (x - K)^+$



Our Pf of BS. is valid (even if g is not diff)

because: We only need Ito on $f(t, S_t)$ with $t < T$

and for $t < T$, f is $C^{1,2}$ (1 der in t & 2 der in x)

Proposition 8.14 (Put call parity). *Consider a European put and European call with the same strike K and maturity T.*

- ▷ $c(t, S_t)$ = *AFP of call (given by (8.5))*
- ▷ $p(t, S_t)$ = *AFP of put.*

Then $\boxed{c(t,x) - p(t,x) = x - Ke^{-r(T-t)}}$, *and hence* $\boxed{p(t,x) = Ke^{-r(T-t)} - x - c(t,x).}$

Consider a put

$\left.\begin{array}{l} +1 \text{ call} \\ -1 \text{ put} \end{array}\right\} \to$ Payoff : $(S_T - K)^+ - (K - S_T)^+$

$= \underbrace{S_T - K}$ ← Forward Contract.

AFP of F.C. $= S_t - Ke^{-r(T-t)}$

$\Rightarrow c(t, S_t) - p(t, S_t) = S_t - Ke^{-r(T-t)}$

8.3. **The Greeks.** Let $c(t,x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

(B.S. Formula.

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad \tau = T - t\,.$$

Definition 8.15. The $\boxed{\textit{delta}}$ is $\partial_x c$.

Remark 8.16 (Delta hedging rule). $\boxed{\Delta_t = \partial_x c(t, S_t).}$

Proposition 8.17. $\partial_x c = N(d_+)$

$$N(y) = \text{CDF of Normal} = \int_{-\infty}^{y} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$c = xN(d_+) - Ke^{-r\tau}N(d_-)$$

$$\partial_x c = N(d_+) + xN'(d_+)\frac{1}{\sigma\sqrt{\tau}\,x} - Ke^{-r\tau}N'(d_-)\frac{1}{\sigma\sqrt{\tau}\,x}$$

$$= N(d_+) + x\left(\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\right)\frac{1}{\sigma\sqrt{\tau}x} - Ke^{-r\tau}\left(\frac{e^{-d_-^2/2}}{\sqrt{2\pi}}\right)\frac{1}{\sigma\sqrt{\tau}x}$$

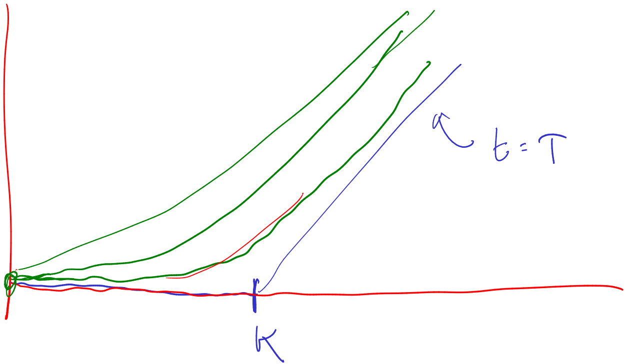
turns out to be 0

set $= N(d_+)$.

Definition 8.18. The *Gamma* is $\partial_x^2 c$ and is given by $\boxed{\partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(\frac{-d_+^2}{2}\right).}$

Definition 8.19. The *Theta* is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)$

$$\partial_x^2 c = \partial_x(\partial_x c) = \partial_x\big(N(d_+)\big) = N'(d_+)\,\frac{1}{\sigma\sqrt{\tau}\,x}$$



$c(t,x)$ as $t \to T$

Proposition 8.20. $\boxed{\textit{(1) } c \textit{ is increasing as a function of } x.}$

(2) c is convex as a function of x.

(3) c is decreasing as a function of t.

(1) $\to$ Since $\partial_x C = N(d_+) > 0$

(2) $\partial_x^2 C =$ Gama > 0

(3) $\partial_t C =$ Theta ≤ 0

Remark 8.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Put Payoff $= (x - K)^+$

R port: Δ_t shares of stock

Rest cash.

Know $\Delta_t = \partial_x c(t, S_t)$

$c(t, S_t) - \Delta_t S_t = c(t, S_t) - \partial_x c(t, S_t) S_t$

Put $x = S_t$

Cash balance $= c(t,x) - x\,\partial_x c(t,x)$

$$= \cancel{x N(d_+)} - K e^{-r\tau} N(d_-) - \cancel{x N(d_+)}$$

$$= -\underline{K e^{-r\tau} N(d_-)} < 0.$$

4

Q: What is Δ_T (for the R part of Eur call)

$$\Delta_T = \begin{cases} 1 & S_T \geq K \\ 0 & S_T \leq K. \end{cases}$$

$x = S_t$.

$$\Delta_t = \partial_x c(t, x) = N(d_+)$$

$\tau = T - t$

$$d_+ = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right)$$

As $t \to T$, $\tau \to 0$

$$\lim_{t \to T} d_+ = \begin{cases} +\infty & x > K \\ -\infty & x < K \end{cases}$$

$$\Rightarrow \lim_{t \to T} \partial_x c = \begin{cases} N(+\infty) = 1 & x > K \\ N(-\infty) = 0 & x < K. \end{cases}$$

Remark 8.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x) -$ tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

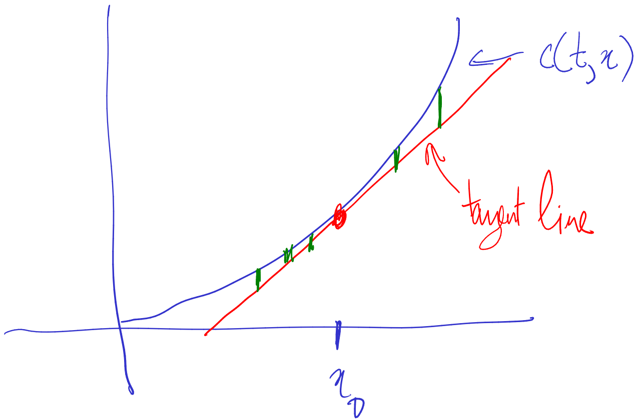
Port value when spot price is x $= c(t,x) + x_0 \partial_x c(t,x_0) - c(t,x_0)$
$- x\, \partial_x c(t,x_0)$

$$= c(t,x) - \Big[\underbrace{(x-x_0)\partial_x c(t,x_0) + c(t,x_0)}_{\text{tangent line}}\Big]$$

u

Portfolio value :

$c(t,x) \longrightarrow$ tgt line



9. Multi-dimensional Itô calculus

- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$\boxed{[X,Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),}$$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X,Y]_t = dX_t \, dY_t$.

QV: $[X, X] = \lim_{\|P\| \to 0} \sum (X_{t_{i+1}} - X_{t_i})^2$

$X \to Y$

$(X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$

Lemma 9.3. $\boxed{[X,Y]_T} = \frac{1}{4}([X+Y,X+Y]_T - [X-Y,X-Y]_T)$

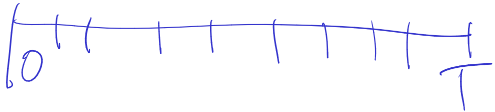
QV of $X+Y$ QV of $X-Y$

$$(a+b)^2-(a-b)^2 = \underline{4ab}.$$

$$\left.\begin{aligned} \Delta_i X &= X_{t_{i+1}} - X_{t_i} \\ \Delta_i Y &= Y_{t_{i+1}} - Y_{t_i} \end{aligned}\right\} \quad \Delta_i X \, \Delta_i Y = \frac{1}{4}\left(\left[\Delta_i (X+Y)\right]^2 - \underline{\left[\Delta_i (X-Y)\right]^2} \right)$$

✓

9. Multi-dimensional Itô calculus



- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),$$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t \, dY_t$.

$$4ab = (a+b)^2 - (a-b)^2$$

Lemma 9.3. $\underbrace{[X, Y]_T}_{\text{Joint QV}} = \frac{1}{4}(\underbrace{[X + Y, X + Y]_T}_{\text{QV of } X+Y} - [X - Y, X - Y]_T)$

QV of $X - Y$

Proposition 9.4. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X, Y]_t = d[M, N]_t$.*

Remark 9.5. Recall, all processes are implicitly assumed to be *adapted* and *continuous*.

Corollary 9.6. *If X is a semi-martingale and B has bounded variation then $[X, B] = 0$.*

1. $$\left.[X,B] = \frac{1}{4}\Big(\underbrace{[X+B,X+B]}_{[X,X]} - \underbrace{[X-B,X-B]}_{[X,X]}\Big)\right\} = 0$$

2. $$\left.\begin{aligned} X &= X_0 + C + M \\ \& \ B &= B_0 + (B-B_0) + 0 \end{aligned}\right\} [X,B] = [M,0] = 0$$

Remark 9.7 (Chain rule). If X, Y are $\boxed{\text{differentiable functions of } t,}$ then

$$d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t$$

Remark 9.8 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_x f = \frac{\partial f}{\partial x}$, $\partial_y f = \frac{\partial f}{\partial y}$.

$$f = f(t, x, y)$$

$$\frac{d}{dt} f(t, X_t, Y_t) \underset{\text{Chain Rule}}{=} \partial_t f(t, X_t, Y_t)\, dt + \left[\partial_x f(\;)\, \frac{dX_t}{\cancel{dt}} + \partial_y f(\;)\, \frac{dY_t}{\cancel{dt}} \right]$$

If X & Y are not diff, then there is an extra term: $\frac{1}{2}\left[\partial_x^2 f\, d[X,X] + \partial_y^2 f\, d[Y,Y] + 2\partial_x \partial_y f\, d[X,Y]\right]$

Theorem 9.9 (Two-dimensional Itô formula).

- *Let X, Y be two processes.*
- *Let $f = f(t, x, y)$ be a $C^{1,2}$ function. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in both x, and y.*
  - ▷ *All the above partial derivatives are continuous. Then:*

from chain rule

$$d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t$$

$2\partial_x\partial_y f$

$$+ \frac{1}{2}\Big[\partial_x^2 f(t, X_t, Y_t)\, d[X, X]_t + \partial_y^2 f(t, X_t, Y_t)\, d[Y, Y]_t + 2\partial_x\partial_y f(t, X_t, Y_t)\, d[X, Y]_t\Big]$$

Remark 9.10. As with the 1D Itô, will drop the arguments (t, X_t, Y_t). *Remember they are there.*

Remark 9.11 (Integral form of Itô's formula).

$$f(T, X_T, Y_T) - f(0, X_0, Y_0) = \int_0^T \partial_t f\, dt + \int_0^T \partial_x f\, dX_t + \int_0^T \partial_y f\, dY_t$$

$$+ \frac{1}{2}\int_0^T \left(\partial_x^2 f\, d[X, X]_t + \partial_y^2 f\, d[Y, Y]_t + 2\partial_x\partial_y f\, d[X, Y]_t\right)$$

Intuition behind Theorem 9.9.

(1) 1D Ito: $\partial_x^2 f \, d[X, X]$ came from taylor

$$f(x+h) = f(x) + h f'(x) + \frac{1}{2} h^2 f''(x) + \text{small}$$

(2) In 2D: Taylors formula is

$$f(x+h, y+k) \approx f(x,y) + h \partial_x f(x,y) + k \partial_y f(x,y)$$

$$+ \frac{1}{2}\Big[\underbrace{\partial_x^2 f \, h^2}_{QV(X)} + \underbrace{\partial_y^2 f \, k^2}_{QV(Y)} + \underbrace{2 \partial_x \partial_y f \, hk}_{\text{Joint QV.}} \Big]$$

Proposition 9.12 (Product rule). $d(XY)_t = X_t\,dY_t + Y_t dX_t + \boxed{d[X,Y]_t}$

If f & g are diff fuc,

$$\frac{d}{\cancel{dt}}(fg) = \frac{df}{\cancel{dt}}\,g + f\,\frac{dg}{\cancel{dt}}$$

Check product rule using Ito!

Let $f(x,y) = xy$

$\partial_t f = 0$

$\partial_x f = y \qquad \partial_y f = x$

$\partial_x^2 f = 0 \qquad \partial_y^2 f = 0$

$\partial_x \partial_y f = 1$

By Itô: $d(X_t Y_t) = d\big(f(X_t, Y_t)\big)$

$$= \partial_t f \, dt + \partial_x f \, dX_t + \partial_y f \, dY_t$$

$$+ \frac{1}{2}\left[\partial_x^2 f \, d[X,X] + \partial_y^2 f \, d[Y,Y] + 2\,\partial_x \partial_y f \, d[X,Y]\right]$$

$$= 0 + Y_t \, dX_t + X_t \, dY_t + \frac{1}{\cancel{2}} \cancel{2} \; 1 \; \underline{d[X,Y]}.$$

$\Rightarrow$ Product Rule.

To use the multi-dimensional Itô formula, we need to compute joint quadratic variations.

Proposition 9.13. *Let M, N be continuous martingales, with* $\boldsymbol{E}M_t^2 < \infty$ *and* $\boldsymbol{E}N_t^2 < \infty$.

(1) *$MN - [M,N]$ is also a continuous martingale.*

(2) *Conversely if $MN - B$ is a continuous martingale for some continuous adapted,* bounded variation *process B with $B_0 = 0$, then $B = [M,N]$.*

Proof.

(1) NTS $MN - [M,N]$ is a mg

$$d\left(MN - [M,N]\right) \overset{\text{Product rule}}{=} M\,dN + N\,dM + d[M,N] - d[M,N]$$

$$= \underbrace{M\,dN + N\,dM}_{Mg}.$$

Proposition 9.14. *(1)* (Symmetry) $[X,Y]=[Y,X]$

(2) (Bi-linearity) *If* $\alpha\in\mathbb{R}$, X,Y,Z *are semi-martingales,* $[X,Y+\alpha Z]=[X,Y]+\alpha[X,Z]$.

Proof.

Proposition 9.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \int_0^t \sigma_s \, dM_s$ *and* $Y_t = \int_0^t \tau_s \, dN_s$.
- *Then* $\boxed{[X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N]_s.}$ ← Gen of our rule to compute QV of Itô int.

Remark 9.16. In differential form, if $dX_t = \sigma_t \, dM_t$ and $dY_t = \tau_t \, dN_t$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

$$\Delta_i X = X_{t_{i+1}} - X_{t_i} \approx \tau_{t_i} \Delta_i M$$

$$\Delta_i Y = \quad \approx \tau_{t_i} \Delta_i N$$

$$\Rightarrow \underbrace{\Delta_i X \, \Delta_i Y}_{d[X, Y]} \approx \tau_{t_i} \tau_{t_i} \underbrace{\Delta_i M \, \Delta_i N}_{\approx d[M, N]}$$

Proposition 9.17. *If M, N are continuous martingales, $\boldsymbol{E}M_t^2 < \infty$, $\boldsymbol{E}N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 9.18 (Warning). Independence implies $\boldsymbol{E}(M_t N_t) = \boldsymbol{E}M_t \boldsymbol{E}N_t$. But it *does not* imply $\boldsymbol{E}_s(M_t N_t) = \boldsymbol{E}_s M_t \boldsymbol{E}_s N_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof. → Tricky

Intuition: $[M,N]_T = \lim_{\|P\| \to 0} \sum \Delta_i M \, \Delta_i N$

Trick: Compute $\boldsymbol{E}\left(\sum \Delta_i M \, \Delta_i N\right)^2 \xrightarrow[\|P\| \to 0]{\text{Will show}} 0$

$$① \; E\left(\sum(\Delta_i M)(\Delta_i N)\right)^2 = E\sum(\Delta_i M)^2(\Delta_i N)^2 + 2E\sum_{i<j}(\Delta_i M)(\Delta_j M)(\Delta_i N)(\Delta_j N)$$

$$= \underbrace{\sum E(\Delta_i M)^2 E(\Delta_i N)^2}_{} + \underbrace{2\sum_{i<j} E(\Delta_i M\,\Delta_j M)\,E(\Delta_i N\,\Delta_j N)}_{\text{Use tower \& } M \text{ is a mg}}$$

check $= 0$

$$\le \left[\max_i E(\Delta_i M)^2\right] \cdot \underbrace{\sum_i E(\Delta_i N)^2}$$

$$\left[\max_i E(\Delta_i M)^2\right] \xrightarrow{\|P\| \to 0} 0$$

$(\because M \text{ is cts!})$

$$\sum_i E(\Delta_i N)^2 \xrightarrow{\|P\| \to 0} E[N, N]$$

Remark 9.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- Let $M_t = \int_0^t \mathbf{1}_{\{W_s < 0\}} \, dW_s$
- Let $N_t = \int_0^t \mathbf{1}_{\{W_s \geqslant 0\}} \, dW_s$

$$M + N = \int_0^t \left(\mathbb{1}_{\{W_s < 0\}} + \mathbb{1}_{\{W_s \geq 0\}} \right) dW$$

$$= \int_0^t 1 \, dW_s = W_t$$

$$d[M,N] = \mathbb{1}_{\{W_s < 0\}} \mathbb{1}_{\{W_s \geq 0\}} \, d[W,W]_t$$

$$= 0 \, dt$$

$$\Rightarrow \boxed{M_t + N_t = W_t}$$

Question 9.20. *Let W^1 and W^2 be two independent Brownian motions, and let $W = (W^1, W^2)$. Define the process X by $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. $\boxed{\text{Is } X \text{ a martingale?}}$*

$W^1_t \longrightarrow 1^{st}$ BM at time t

$W^2_t \longrightarrow 2^{nd}$ BM at time t

$W = (W^1, W^2)$ $\quad$ $|W| = \sqrt{(W^1)^2 + (W^2)^2}$

$X_t = \ln\left[(W^1_t)^2 + (W^2_t)^2\right]$

Is X a mg?

Itô: $f(t,x,y) = \ln(x^2+y^2)$

Compute $\partial_x f$, $\partial_y f$, $\partial_x^2 \cdots$ ...

Compute $dX_t = d f(t, W_t^1, W_t^2)$

$$\stackrel{\text{Itô}}{=} \partial_x f \, dW_t^1 + \partial_y f \, dW_t^2 + \frac{1}{2}\Big[\partial_x^2 f \, \underbrace{d[W^1,W^1]}_{dt} + \partial_y^2 f \, \underbrace{d[W^2,W^2]_t}_{dt} + 2\partial_x\partial_y f \, d[W^1,W^2]\Big]$$

$\sim \partial$ (indep)

$$= \partial_x f \, dW_t^1 + \partial_y f \, dW_t^2 +$$

$$+ \frac{1}{2} \left[\partial_x^2 f + \partial_y^2 f \right] dt$$

Can compute & check $\underline{\partial_x^2 f + \partial_y^2 f} = 0$ (when $f(x,y) = \ln(x^2+y^2)$)

$$\boxed{\Rightarrow dX_t = \partial_x f \, dW_t^1 + \partial_y f \, dW_t^2} \Rightarrow X \text{ is a mg ??}$$

Claim: Even though there are no dt terms here X is NOT a mg!!

(Reason: ______________________)

10. Risk Neutral Pricing

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $\boxed{D_t = \exp\Big(-\int_0^t R_s \, ds\Big)}$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

Recall: C_t = cash in bank at time t

$$\partial_t C_t = R_t C_t \implies C_t = C_0 \underbrace{\exp\Big(\int_0^t R_s \, ds\Big)}$$

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P},}$ *where*

Market Price of risk

$$\to \quad Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}.}$$

Theorem 10.5. *Any security can be replicated. If a security pays* V_T *at time* T*, then the arbitrage free price at time* t *is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s \, ds\Big) V_T\Big).$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Disc time: RNP formula $\quad V_n = \frac{1}{D_n} \tilde{E}_n(D_N V_N)$

(disc time version of

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$.

(Need $Z_T > 0$ to ensure
$P(A) = 0 \iff \tilde{P}(A) = 0$

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then $\tilde{\boldsymbol{E}}_s X_t = \frac{1}{Z_s} \boldsymbol{E}_s (Z_t X_t)$.*

RN cond Exp.

Proof. You will see this in the proof of the Girsanov theorem. $\square$

Corollary 10.11. *M is martingale under $\tilde{\boldsymbol{P}}$ if and only if ZM is a martingale under $\boldsymbol{P}$.*

Q: $\tilde{P}(A) \in [0,1]$

Check: ① $\tilde{P}(A) = E(\mathbb{1}_A Z_T) \geq 0 \quad (\because Z_T > 0)$

② $\tilde{P}(A) = E(\mathbb{1}_A Z_T) \leq E Z_T = 1$

Q: Need $\tilde{P}(\Omega) = 1$

1 Note $\tilde{P}(\Omega) = E(\mathbb{1}_\Omega Z_T) = E Z_T = 1$

Prop: $X_t \to \mathcal{F}_t$ meas

$$Z_t = E_t Z_T$$

Then $\tilde{E}_s X_t = \frac{1}{Z_s} E_s(Z_t X_t)$

Claim: M is a $\tilde{P}$ mg $\iff$ ZM is a P mg.

① Say ZM is a P-mg

Compute $\tilde{E}_s M_t = \frac{1}{Z_s} E_s(Z_t M_t)$

$= \frac{1}{Z_s}(Z_s M_s)$ ($\because ZM$ is a P-mg)

$= M_s \Rightarrow M$ is a $\tilde{P}$ mg.

Last time:

Market

Generalized GBM (α, σ adapted processes)

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$$

Discount factor $D_t = \exp\left(-\int_0^t R_s \, ds\right)$

Interest rate R_t (adapted process)

C_t = cash in Bank at time t

$$dC_t = +R_t C_t \, dt \quad \left(C_t = C_0 \exp\left(+\int_0^t R_s \, ds\right)\right)$$

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P},}$ *where*

IOU explation $\rightarrow$ $$Z_T = \exp\Big(-\int_0^T \theta_t\, dW_t - \frac{1}{2}\int_0^T \theta_t^2\, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Theorem 10.5. *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\, ds\Big) V_T\Big).$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$ and prove both the above theorems later.

Note $$\frac{1}{D_t}\tilde{E}_t(D_T V_T) = \frac{1}{D_t}\tilde{E}_t\Big(\exp\Big(-\int_0^T R_s\, ds\Big) V_T\Big)$$

$$= \frac{1}{D_t}\tilde{E}_t\Big(\underbrace{\exp\Big(-\int_0^t R_s\, ds\Big)}\ \exp\Big(-\int_t^T R_s\, ds\Big) V_T\Big) =$$

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$.

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then* $\boxed{\tilde{\boldsymbol{E}}_s X_t = \dfrac{1}{Z_s} \boldsymbol{E}_s (Z_t X_t)}$.

Proof. You will see this in the proof of the Girsanov theorem. □

Corollary 10.11. *M is martingale under $\tilde{\boldsymbol{P}}$ if and only if $\boxed{ZM}$ is a martingale under $\boldsymbol{P}$.*

↳ Check: Say M is a $\tilde{P}$ mg.

Want to show ZM is a P mg.

Compute $E_s(Z_t M_t) = Z_s \tilde{E}_s M_t$ (Lemma 10.10)

$= Z_s M_s$ ($\because$ M is a $\tilde{P}$ mg)

Conversely, say ZM is a P mg.

NTS M is a $\tilde{P}$ mg: $\tilde{E}_s M_t = \frac{1}{Z} E_s(Z_t M_t) \overset{\text{mg}}{=} \frac{1}{\cancel{Z_s}} \cancel{Z_s} M_s$

Theorem 10.12 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$ and let b be an adapted process.*

- *Define* $\tilde{W}_t = W_t + \int_0^t b_s\, ds$ *(i.e.* $d\tilde{W}_t = b_t\, dt + dW_t$*).*
- $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$, *where* $Z_t = \exp\Big(-\int_0^t b_s\, dW_s - \frac{1}{2}\int_0^t |b_s|^2\, ds\Big)$.

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

$d\tilde{W} = b\, dt + dW$

Proposition 10.13. $dZ_t = -Z_t b_t \cdot dW_t$.

Question 10.14. *Looks like Z is a martingale. Why did we assume it in Theorem 10.12?*

Let $X_t = \int_0^t b_s \, dW_s \longrightarrow d[X,X] = b_t^2 \, dt$

Let $f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t b_s^2 \, ds\right)$

Thus $Z_t = f(t, X_t)$

$$\left|\begin{array}{l} \partial_t f = \exp(\quad)\left(-\frac{1}{2} b_t^2\right) \\ \partial_x f = -\exp(\quad) \\ \partial_x^2 f = +\exp(\quad) \end{array}\right.$$

Ito

$\downarrow$

$$dZ = d\,f(t, X_t) = \partial_t f\,dt + \partial_x f\,dX + \tfrac{1}{2}\partial_x^2 f\,d[X,X]$$

$$= -\tfrac{1}{2} b_t^2 Z_t\,dt - Z_t\,dX_t + \tfrac{1}{2} Z_t\,d[X,X]$$

$$= \cancel{-\tfrac{1}{2} b_t^2 Z_t\,dt} - Z_t b_t\,dW + \cancel{\tfrac{1}{2} Z_t b_t^2\,dt}$$

$$= \underline{-Z_t b_t\,dW_t}$$

Note: $\int_0^t \sigma_s \, dW_s$ is a mg ONLY if $\boxed{E\int_0^t \sigma_s^2 \, ds < \infty}$

Even though $dZ_t = -b_t Z_t \, dW_t$

Z is only a mg if $E\int_0^t b_s^2 Z_s^2 \, ds < \infty$

(Not easy to check in general)

Idea behind the proof of Theorem 10.12.

To show $\tilde{W}$ is a BM under $\tilde{P}$

Use Levy's Criterion

$\tilde{W}$ is a BM under $\tilde{P}$ $\Longleftrightarrow$ 0 $\tilde{W}$ is a cts process

1 $\tilde{W}$ is a mg

& 2 $[\tilde{W}, \tilde{W}]_t = t$

Check 1 & 2 :

$$d\tilde{W} = \underbrace{b\,dt}_{\text{does not affect QV}} + dW \quad \Big\} \Rightarrow d[\tilde{W},\tilde{W}] = dt \Rightarrow (2).$$

Check (1): Check $\tilde{W}$ is a mg under $\tilde{P}$

$\iff Z\tilde{W}$ is a mg under P.

By Product rule, $d(Z\tilde{W}) = Z\,d\tilde{W} + \tilde{W}\,dZ + d[Z,\tilde{W}]$

Recall $\boxed{dZ = -b Z_t\, dW}$ & $d\tilde{W} = b\,dt + dW$

$$\Rightarrow d(Z\tilde{W}) = Z_t(b_t\,dt + dW_t) - \tilde{W}_t b_t Z\,dW_t + \overbrace{(-b_t Z_t \cdot 1)\,dt}$$

$$= Z(1 - \tilde{W} b)\,dW_t$$

$\Rightarrow Z\tilde{W}$ is a mg. under P $\Rightarrow$ (1)

By hung $\Rightarrow$ $\boxed{\tilde{W} \text{ is a mg under } \tilde{P}}$

Theorem (Theorem 10.4). *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P},}$ *where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Proof of Theorem 10.4.

Want $D_t S_t$ to be a $\tilde{P}$ mg.

$D_t = \exp\left(-\int_0^t R_s \, ds\right) \Rightarrow dD_t = -R_t D_t \, dt$ (Finite 1st var)

$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW$

Compute $d(D_t S_t) = D_t \, dS_t + S_t \, dD_t + \overbrace{d[D_t, S_t]}^{0}$

$$= D_t\left(\alpha_t S_t\,dt + \sigma_t S_t\,dW\right) - R_t D_t S_t\,dt + 0$$

$$= D_t S_t(\alpha_t - R_t)\,dt + D_t\sigma_t S_t\,dW_t$$

$$= D_t\sigma_t S_t\left[\underbrace{\frac{\alpha_t - R_t}{\sigma_t}\,dt + dW}_{d\widetilde{W}}\right]$$

Choose $\widetilde{W} = W_t + \int_0^t \Theta_s\,ds$, where $\Theta_t = \dfrac{\alpha_t - R_t}{\sigma_t}$.

Choose $\tilde{P}$ by Girsanov to make $\tilde{W}$ a BM.

$$\Rightarrow d\tilde{P} = Z_T \, dP \,, \quad Z_T = \exp\left(-\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right)$$

$$\Rightarrow d(D_t S_t) = D_t S_t \sigma_t \, \underbrace{d\tilde{W}}_{\text{BM under } \tilde{P}}$$

$\Rightarrow D_t S_t$ is a mg under $\tilde{P}$. !!!

Theorem 10.15. *X_t represents the wealth of a self-financing portfolio if and only if $D_t X_t$ is a $\tilde{\boldsymbol{P}}$ martingale.*

Remark 10.16. The proof of the backward direction requires the *martingale representation theorem*, and is outlined on your homework.

Remark 10.17. This is the analog of Theorem 4.57

Proof of the forward direction.

Assume X is a self fin port

Show $D_t X_t$ is a $\tilde{P}$ mg.

(1) Self fin $\Rightarrow$ $dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t) \, dt$

(2) Compute dS in terms of $d\tilde{W}$

$$d\tilde{W}_t = \theta_t \, dt + dW \qquad \theta = \frac{\alpha - R}{\sigma}$$

$$dS = \alpha S \, dt + \sigma S \, dW$$

$$= \alpha S \, dt + \sigma S \left(d\tilde{W} - \theta \, dt \right)$$

$$= \alpha S \, dt + \sigma S \, d\tilde{W} - (\alpha - R) S \, dt$$

$$\boxed{dS = R S \, dt + \sigma S \, d\tilde{W}}$$

Replaced $W \longrightarrow \tilde{W}$

& $\alpha \longrightarrow R$

Compute $d(D_t X_t) = D_t\, dX_t + X_t\, dD_t + d[D_t, X_t]$

$$= D_t\left(\Delta_t\, dS_t + R(X - \Delta_t S_t)\,dt\right) - R D_t X_t\, dt$$

$$= D_t \Delta_t \left(R_t S_t\, dt + \sigma_t S_t\, d\tilde{W}\right) - D_t \Delta_t R_t S_t\, dt$$

$$= D_t \Delta_t \sigma_t S_t\, d\tilde{W} \quad \Rightarrow \quad D_t X_t \text{ is a } \tilde{P} \text{ mg!!}$$

Theorem (Theorem 10.5). *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\,ds\Big)\,V_T)\Big)\,.$$

Remark 10.18. This is the analog of Proposition 4.1.

Proof of Theorem 10.5.

Replicate the sec.: Find a self fin port with payoff $= V_T$

Choose $X_t = \dfrac{1}{D_t} \tilde{E}_t(D_T V_T)$.

1. NTS $X_T = V_T$ (true)

(2) NTS X_t = wealth of a self fin port.

$\iff D_t X_t$ is a $\tilde{P}$ mg.

Formula for X_t

Compute $\tilde{E}_s(D_t X_t) = \tilde{E}_s\left(\tilde{E}_t(D_T V_T)\right)$

$\underline{\underline{\text{tower}}}\ \tilde{E}_s(D_T V_T)$

$= D_s X_s$ (Formula for X_s).

11. Black Scholes Formula revisited

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

Theorem 11.1. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}, \qquad \tau = T - t. \tag{8.4}$$

Remark 11.2. This proves Proposition 8.8.

$D_t = e^{-rt}$

Proof: RNP Formula: $V_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$

$= e^{-r\tau} \tilde{E}_t\, g(S_T) \leftarrow (*)$

Under $\tilde{P}$: $dS_t = r S_t\, dt + \sigma S_t\, d\tilde{W}_t$

$\Rightarrow S$ is $GBM(r, \sigma)$ under $\tilde{P}$.

$$\Rightarrow S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right)$$

$$\Rightarrow S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T\right)$$

$$\Rightarrow S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

Substitute in $(*)$

$$V_t = e^{-r\tau}\, \tilde{E}_t\, g(S_T)$$

$$= e^{-r\tau}\, \tilde{E}_t\, g\Big[\underbrace{S_t}_{\mathcal{F}_t \text{ meas}} \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, \overbrace{\frac{(\tilde{W}_T - \tilde{W}_t)}{\sqrt{\tau}}}^{N(0,\,T-t)\ \&\ \mathcal{F}_t \text{ indep}}\Big)\Big]$$

<u>Indep lemma</u>

$$(**) \rightarrow e^{-r\tau} \int_{-\infty}^{\infty} g\Big(S_t \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}\, dy}{\sqrt{2\pi}}$$

QED!!

Theorem 11.3 (Black Scholes Formula). *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

is the CDF of a standard normal variable.

Remark 11.4. This proves Corollary 8.9.

Substitute $g(x) = (x-K)^+$ in $(**)$

$$c(t,x) = e^{-r\tau}\int_{-\infty}^{\infty}\left(x\,e^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}y} - K\right)^+ e^{-y^2/2}\,\frac{dy}{\sqrt{2\pi}} \quad (***)$$

$$S_{\text{alive}} \times e^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}\, y} = K$$

$$\Rightarrow (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y = \ln\left(\frac{K}{X}\right) = -\ln\left(\frac{X}{K}\right)$$

$$\Rightarrow y = -\frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{X}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right)$$

$$= -d_-$$

$$\Rightarrow (***)$$

$$C(t,x) = e^{-r\tau} \int_{-d_-}^{\infty} \left(x e^{(r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y} - K \right) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

& simplify ...

① FCE's $\longrightarrow$ 75% (Pittsburgh)

$\longrightarrow$ < 75% (NY $\longrightarrow$)

② Final $\longrightarrow$ Practice: $\longrightarrow$ Do problems yourself

$\longrightarrow$ ⓐ some perils $\longrightarrow$ standard (Basic tools)

ⓑ little $\longrightarrow$ "more involved problems"

ⓒ Very little $\longrightarrow$ "out of box" thinking

12. **Review problems**

Problem 12.1. Consider a financial market consisting of a risky asset and a money market account. Suppose the return rate on the money market account is r, and the price of the risky asset, denoted by S, is a geometric Brownian motion with mean return rate α and volatility σ. Here r, α and σ are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$\boxed{V_T = \frac{1}{T} \int_0^T S_t \, dt}$$

at maturity T. Also compute the trading strategy in the replicating portfolio.

$S = GBM(\alpha, \sigma)$

① Price security:

$$V_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$$

$$D_t = \exp\left(-\int_0^t R_s \, ds\right) = e^{-rt}$$

$$\frac{D_T}{D_t} = e^{-r(T-t)}$$

$$(\tau = T - t)$$

$$\Rightarrow V_T = \frac{e^{-r\tau}}{T} \tilde{E}_t \int_0^T S_s \, ds$$

$$= \frac{e^{-r\tau}}{T} \left(\int_0^t S_s \, ds + \int_t^T \tilde{E}_t S_s \, ds \right)$$

$$\underbrace{\tilde{E}_t \int_0^T \sigma_s \, d\tilde{W}_s}_{Mg} = \int_0^t \sigma_s \, d\tilde{W}_s$$

$$\tilde{E}_t \int_0^T b_s \, ds = \int_0^T \tilde{E}_t b_s \, ds$$

$$= \int_0^t b_s \, ds + \int_t^T \tilde{E}_t b_s \, ds$$

Need to compute $\tilde{E}_t S_s$ when $s > t$.

Option 1: Under $\tilde{P}$, $S = GBM(r, \sigma)$

$$\boxed{S_s = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)s + \sigma \tilde{W}_s\right)}$$

$\tilde{W}$ is a BM under $\tilde{P}$.

Indep lemma & compute $\tilde{E}_t(\ )$.

Better way Option 2:

Under $\tilde{P}$, $e^{-rs} S_s$ is a $\tilde{P}$ mg

$$\therefore \quad \tilde{E}_t(S_s) = e^{rs}\,\tilde{E}_t\big(\underbrace{e^{-rs} S_s}_{\tilde{P}\text{ - mg}}\big) \qquad (s > t)$$

$$= e^{rs}\, e^{-rt} S_t = S_t\, e^{-r(t-s)}$$

Substitute back:

$$V_t = \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + \int_t^T \tilde{E}_t S_s\,ds\right)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + S_t \int_t^T e^{r(s-t)}\,ds\right)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + \frac{S_t}{r}\left[e^{r(T-t)} - 1\right]\right)$$

Q: Trading Strategy?? $\Delta_t = ??$

Note: If Payoff $= g(S_T)$

Delta Hedging — Then $V_t = f(t, S_t)$ & $\Delta_t = \partial_x f(t, S_t)$

Let X_t = wealth of R. port. $(X_t = V_t)$

Know $dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt$

$$= \left(\cancel{\Delta_t rS} - r\cancel{\Delta_t S} + rX_t \right) dt + \Delta_t \sigma S_t \, d\tilde{W}$$

$$\boxed{dS = rS\,dt + \sigma S\,d\tilde{W}}$$

Use formula for V & find coeff of $d\tilde{W}$

$$V_t \overset{Le}{=} \frac{e^{-rT}}{T} \left(\int_0^t S_s\,ds + \frac{S_t}{r} \left[e^{r(T-t)} - 1 \right] \right)$$

Want this

$$\Rightarrow dV_t = (\text{uuu}) \, dt + (\quad) \, d\tilde{W}$$

$$dV_t = d\left(e^{(\,)}\int_0^t \quad ds\right) + d\left[\frac{S_t}{r}\left(e^{r(T-t)}-1\right)\cdot\frac{e^{-rT}}{T}\right]$$

$$\underset{0}{}$$

$$= \left(m\,dt + S_t\,d(\,) + \frac{1-e^{-rT}}{rT}\,dS_t + d[\,,\,]\right)$$

$$= (\text{smthg})\,dt + \boxed{\frac{1-e^{-rT}}{rT}\,\sigma S_t\,d\tilde{W}}$$

a

equate this to $\Delta_t \sigma S_t$.

$$\Rightarrow \Delta_t = \boxed{\frac{1 - e^{-r\tau}}{r\tau}}$$

Problem 12.2. Let $X \sim N(0,1)$, and $a, \alpha, \beta \in \mathbb{R}$. Define a new measure $\tilde{\boldsymbol{P}}$ by

$$\boxed{d\tilde{\boldsymbol{P}} = \exp\big(\alpha X + \beta\big)\, d\boldsymbol{P}\,.}$$

Find α, β such that $X + a \sim N(0,1)$ under $\tilde{\boldsymbol{P}}$.

Sol: Compute $\tilde{E}\left(e^{\lambda(X+a)}\right) \overset{\text{Want}}{=} e^{\lambda^2/2}$

$$\tilde{E}\, e^{\lambda(X+a)} = \int_\Omega e^{\lambda(X+a)}\, d\tilde{P} = \int_\Omega e^{\lambda(X+a)}\, e^{(\alpha X+\beta)}\, dP$$

$$= E\left(e^{\lambda(X+a)}\, e^{(\alpha X+\beta)}\right)$$

$$= E\, e^{(\lambda+\alpha)X + \lambda a + \beta}$$

$$= e^{\lambda a + \beta + (\lambda+\alpha)^2/2} \quad \underline{\underline{\text{Want}}} \quad e^{\lambda^2/2}.$$

Equate $\boxed{\lambda a + \beta + \frac{(\lambda+\alpha)^2}{2} = \frac{\lambda^2}{2} \text{ \& solve.}}$

Problem 12.3. Let f be a deterministic function, and define

$$X_t \stackrel{\text{def}}{=} \int_0^t f(s) W_s \, ds \, .$$

Find the distribution of X.

Guess: $\displaystyle\int_0^t f(s) W_s \, ds \sim N$

Reason: $\displaystyle\int_0^t f(s) W_s \, ds = \lim_{\|P\| \to 0} \underbrace{\sum \underbrace{f(t_i)}_{\text{not random}} W_{t_i} \underbrace{(t_{i+1} - t_i)}_{\text{not random}}}_{\text{linear comb of a Joint normal.}}$

Normal!! ←

Need to find EX_t & $\operatorname{Var} X_t$

① $EX_t = E\int_0^t f(s) W_s \, ds = \int_0^t f(s) \, EW_t \, ds = 0$

② $EX_t^2 = E\left(\int_0^t f(s) W_s \, \underline{\underline{ds}}\right)^2 =$

$= E\left(\int_0^t f(s) W_s \, ds\right)\left(\int_0^t f(\cancel{s}^{\,r}) W_{\cancel{s}_{\,r}} \, d\cancel{s}_{\,r}\right)$

Do Isom:

$$E\left(\int_0^t \sigma_s \, dW_s\right)^2 = E\int_0^t \sigma_s^2 \, ds$$

$$= E \int_{s=0}^{t} \int_{r=0}^{t} f(s)\, f(r)\, W_s W_r \, dr \, ds$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} f(s)\, f(r)\, \underbrace{E(W_s W_r)}_{s \wedge r \ (s \text{ min } r) \ \& \text{ compute.}} \, dr \, ds$$

Problem 12.4. Let $x_0, \mu, \theta, \sigma \in \mathbb{R}$, and suppose X is an Itô process that satisfies

$$dX(t) = \theta(\mu - X_t)\,dt + \sigma\,dW_t\,,$$

(O. U. Process)

with $X_0 = x_0$.

(a) Find functions $f = f(t)$ and $g = g(s,t)$ such that

$$\boxed{X(t) = f(t) + \int_0^t g(s,t)\,dW_s\,.}$$

The functions f, g may depend on the parameters x_0, θ, μ and σ, but should not depend on X.

(b) Compute $\boldsymbol{E}X_t$ and $\operatorname{cov}(X_s, X_t)$ explicitly.

(a) Compute $d\left(e^{\theta t} X_t\right) = e^{\theta t} dX + X_t\,\theta e^{\theta t} dt + \underbrace{d[e^{\theta t}, X]_t}_{0}$

$= e^{\theta t}\left(\theta(\mu - X)dt + \sigma\,dW\right) + X\,\theta\,e^{\theta t}\,dt$

$= e^{\theta t}\theta\mu\,dt + e^{\theta t}\sigma\,dW_t$

$$\Rightarrow e^{\theta t} X_t - X_0 = \mu \int_0^t \theta e^{\theta s}\, ds + \int_0^t \sigma e^{\theta s}\, dW_s$$

$$= \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s}\, dW_s$$

$$\Rightarrow X_t = \underbrace{e^{-\theta t} X_0 + \mu\left(1 - e^{-\theta t}\right)}_{EX_t} + \sigma\, \underbrace{e^{-\theta t} \int_0^t e^{\theta s}\, dW_s}_{E=0}$$

(b) Compute $\mathrm{Cov}(X_s, X_t)$

$$\mathrm{Cov}(X_s, X_t) = E(X_s - EX_s)(X_t - EX_t)$$

$$= E\left(\sigma e^{-\theta s}\int_0^s e^{\theta r}\,dW_r\right)\left(\sigma e^{-\theta t}\int_0^t e^{\theta r}\,dW_r\right)$$

$$= \sigma^2 e^{-\theta(s+t)}\, E\left(\int_0^s e^{\theta r}\,dW_r \int_0^t e^{\theta r}\,dW_r\right)$$

$$= \sigma^2 e^{-\theta(s+t)}\, E\left(\int_0^s e^{\theta r}\,dW_r \left(\int_0^s e^{\theta r}\,dW_r + \int_s^t e^{\theta r}\,dW_r\right)\right)$$

$$= \sigma^2 e^{-\theta(s+t)} \left[E\left(\int_0^s e^{\theta r}\, dW_r \right)^2 + E\left[\underbrace{\int_0^s e^{\theta r}\, dW_r}_{\mathcal{F}_s \text{ meas}} \underbrace{\int_s^t \underbrace{e^{\theta r}}_{\text{not random}} \underbrace{dW_r}_{\mathcal{F}_s\text{-indep}}}_{\text{Indep of } \mathcal{F}_s} \right] \right]$$

$$= \sigma^2 e^{-\theta(s+t)} \left[\cancel{E} \int_0^s e^{2\theta r}\, dr + 0 \right]$$

& compute.

Problem 12.5. Let W be a Brownian motion, and define

$$B_t = \int_0^t \operatorname{sign}(W_s)\, dW_s \,.$$

$$\operatorname{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(a) Show that B is a Brownian motion.

(b) Is there an adapted process σ such that

$$W_t = \int_0^t \sigma_s \, dB_s \,?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation $[B, W]$.

(d) Are B and W uncorrelated? Are they independent? Justify.

(a) : Levy's Criterion : Need ① X is a cts Martingale

② $[X, X]_t = t$

① X is a cts mg since $dX_t = (\ \ ) dW_t$ (Ito int is a mg)

(2) $d[X,X]_t = \text{sign}(W_t)^2\, dt = 1\, dt$

$\Rightarrow$ (a).

(b) $dB_s = \text{sign}(W_s)\, dW_s \Rightarrow dW_s = \dfrac{1}{\text{sign}(W_s)}\, dB_s = \text{sign}(W_s)\, dB_s$

$$\Rightarrow W_t = \int_0^t \text{sign}(W_s)\, dB_s$$

(c) Compute $[B, W]$:

$$dB_t = \mathrm{sign}(W_t)\, dW_t$$
$$dW_t = 1 \quad dW_t$$

$$d[B, W] = \mathrm{Sign}(W_t) \cdot 1 \cdot d[W_t, W_t]$$
$$= \mathrm{Sign}(W_t)\, dt$$

(d). Are B & W uncor?

$$\left.\begin{aligned} EB_t &= 0 \\ EW_t &= 0 \end{aligned}\right\}$$

Compute $E(B_t W_t)$

$\underbrace{\hspace{6em}}_{u}$

$$d(B_t W_t) = B_t \, dW_t + W_t \, dB_t + d[B, W]_t$$

$$= B_t \, dW_t + W_t \, \mathrm{sign}(W_t) \, dW_t + \mathrm{Sign}(W_t) \, dt$$

$$\Rightarrow B_t W_t - \underbrace{B_0 W_0}_{0} = \int_0^t \left(B_s + W_s \, \mathrm{sign}(W_s) \right) dW_s + \int_0^t \mathrm{Sign}(W_s) \, ds.$$

$$\Rightarrow E(B_t W_t) = E \int_0^t (\quad) \, dW_s + E \int_0^t \mathrm{sign}(W_s) \, ds$$

$$= 0 + \int_0^t \underbrace{\mathbb{E}\,\operatorname{sign}(W_s)}_{0}\, ds$$

$$\Rightarrow \mathbb{E}(B_t W_t) = 0 \quad \Rightarrow B_t \;\&\; W_t \text{ are uncor!!}$$

Q: Are B_t & W_t indep?

B_t & W_t are Nomal & uncor

$\not\Rightarrow$ Indep (Not Jointly normal).

B & W are NOT indep

because If B & W were indep then $[B, W] = 0$

But $d[B, W]_t = \operatorname{Sign}(W_t)\, dt \neq 0$.

Problem 12.6. Suppose σ, τ, ρ are three deterministic functions and M and N are two continuous martingales with respect to a common filtration $\{\mathcal{F}_t\}$ such that $M_0 = N_0 = 0$, and

$$d[M, M]_t = \sigma_t \, dt \,, \quad d[N, N]_t = \tau_t \, dt \,, \qquad \text{and} \qquad d[M, N]_t = \rho_t \, dt \,.$$

(a) Compute the joint moment generating function $\boldsymbol{E} \exp(\lambda M(t) + \mu N(t))$.
(b) *(Lévy's criterion)* If $\sigma = \tau = 1$ and $\rho = 0$, show that (M, N) is a two dimensional Brownian motion.

Problem 12.7. Let W be a Brownian motion. Does there exist an equivalent measure $\tilde{\boldsymbol{P}}$ under which the process tW_t is a Brownian motion? Prove it.

Problem 12.8. Let $\theta \in \mathbb{R}$ and define

$$Z_t = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big) .$$

Given $0 \leqslant s < t$, and a function f, find a function such that

$$\boldsymbol{E}_s f(Z_t) = g(Z_s) .$$

Your formula for the function g can involve f, s, t and integrals, but not the process Z or expectations.

Problem 12.9. Consider the N period Binomial model with $N = 5$, and parameters $0 < d < 1 + r < u$. At maturity $N = 5$, a security pays \$1 if $S_5 > (1 + r)S_4$, and 0 otherwise. Find the arbitrage free price and trading strategy trading at time 0.