## Stochastic Calculus for Finance I: Midterm.

2022-11-15, Pittsburgh

- This is a closed book test. You may use a calculator. You may not give or receive assistance.
- Your calculator must not be able to access the internet, or store/read document files (PDF, word, etc.)
- You have 90 minutes. The exam has a total of 5 questions and 25 points.
- The questions are roughly ordered by difficulty. Good luck.

In this exam W always denotes a standard Brownian motion, and the filtration  $\{\mathcal{F}_t | t \ge 0\}$  (if not otherwise specified) is the Brownian filtration.

- 5 1. Let  $X_t = W_t^3 e^{2t}$ . Express X as the sum of a martingale and a process of finite first variation.
- 5 2. Let X be a random variable that is determined by the flip of a fair coin, with X = 1 when the coin lands heads and X = 0 when the coin lands tails. Let Y be a standard normal that is independent of X. Let  $\lambda \in \mathbb{R}$ . Compute  $\mathbf{E}(e^{\lambda XY} | X)$ . Express your answer in the form g(X, Y) for some function g that you find an explicit formula for. (Your formula for g should not involve expectations, conditional expectations, or integrals.)

5 3. Let 
$$X_t = W_t^2 + \int_0^t \sqrt{1+s^2} \, dW_s + \int_0^t \sqrt{3+W_s^4} \, ds$$
. Find the quadratic variation of X. Express your answer as 
$$\int_0^t f(s, W_s) \, ds + \int_0^t g(s, W_s) \, dW_s \,,$$

for some functions f and g you explicitly compute a formula for.

5 4. Consider a discrete time market consisting of a bank and a stock. Let  $S_n$  denote the stock price at time n, and we know  $S_0 = \$20$ . The stock price changes according to the flip of a biased coin that lands heads with probability 1/3 and tails with probability 2/3. If the coin lands heads the stock price increases by 10% (i.e.  $S_{n+1} = 1.1S_n$ ), and if the coin lands tails the stock price increases by 5% (i.e.  $S_{n+1} = 1.05S_n$ ). The interest rate in the bank is initially 6%. After the first coin flip, the interest rate becomes 7%.

Consider a European call option with strike price \$22.50 that matures at time N = 2. That is, at time n = 2, this option pays the holder  $S_2 - 22.50$  dollars if  $S_2 > 22.50$ , and pays nothing otherwise. Find the arbitrage free price of this option at time n = 0. Also find the number of shares held in the replicating portfolio at time n = 0. Round your final answer two decimal places. (I recommend rounding intermediate steps to three decimal places.)

5 5. Let 
$$X_t = \int_0^t e^{-W_r} dW_r$$
. Given  $0 \le s < t$ , find  $E_s[X_t^2]$ . Express your answer in the form

$$\boldsymbol{E}_{s}[X_{t}^{2}] = \int_{0}^{s} f(r, X_{r}, W_{r}) \, dr + \int_{0}^{s} g(r, X_{r}, W_{r}) \, dW_{r} + h(s, t, W_{s}, X_{s})$$

for some functions f, g and h that you explicitly find a formula for.