12.9

$$V_0 = \frac{1}{(1+r)^5} \widetilde{\mathbb{E}}[V_5]$$ ✓

$$V_4 = \frac{1}{1+r} \widetilde{\mathbb{E}}_4[V_5]$$

$$V_5 = \mathbb{I}(S_5 > (1+r) S_4)$$

$$\Delta_0 = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$S_5 = \begin{cases} u S_4 & H \\ d S_4 & T \end{cases}$$

$$\Delta 4 \neq 0$$

$$V_5 = \begin{cases} 1 & \omega_5 = H \\ 0 & \omega_5 = T \end{cases}$$

$$V_0 = \frac{1}{(1+r)^5} \widetilde{P}(\omega_5 = H)$$

2021 #2.

$$\left(\log\left(\frac{S_{n+1}}{S_n}\right)\right)^2 = \begin{cases} (\log u)^2 & \omega_{n+1} = H \\ (\log d)^2 & \omega_{n+1} = T \end{cases}$$

$$\widetilde{\mathbb{E}}\left[\left(\log\left(\frac{S_{n+1}}{S_n}\right)\right)^2\right] = (\log u)^2 \cdot \tilde{p} + (\log d)^2 \cdot \tilde{q}$$

12.6

(a) $X_t = e^{\lambda M_t + \mu N_t}$

$$dX_t = \underline{\lambda X_t \, dM_t} + \underline{\mu X_t \, dN_t}$$
$$+ \frac{1}{2}\lambda^2 X_t \, d\langle M, M\rangle_t$$
$$+ \frac{1}{2}\mu^2 X_t \, d\langle N, N\rangle_t$$
$$+ \lambda\mu X_t \, d\langle M, N\rangle_t$$

$$= \text{martingale} + \left(\frac{1}{2}\lambda^2 \sigma_t + \frac{1}{2}\mu^2 \tau_t + \lambda\mu\rho_t\right) X_t \, dt$$

$$\mathbb{E}[X_t] = 1 + \int_0^t \left(\frac{1}{2}\lambda^2 \sigma_s + \frac{1}{2}\mu^2 \tau_s + \lambda\mu\rho_s\right)\mathbb{E}[X_s]\, ds$$

$f(t) = \mathbb{E}[X_t]$, $f(0) = 1$

$$f'(t) = \left(\frac{1}{2}\lambda^2 \sigma_t + \frac{1}{2}\mu^2 \tau_t + \lambda\mu\rho_t\right) f(t)$$

$$\Rightarrow f(t) = e^{\int_0^t \frac{1}{2}\lambda^2\sigma_s + \frac{1}{2}\mu^2\tau_s + \lambda\mu\rho_s \, ds}$$

(6) $\sigma = \tau = 1$, $\rho = 0$

$$\underline{f(t)} = e^{\int_0^t \frac{1}{2}\lambda^2 + \frac{1}{2}\mu^2 \, ds} = e^{\frac{1}{2}\lambda^2 t + \frac{1}{2}\mu^2 t}$$

$$\rotatebox{90}{=}$$

$$E\left[e^{\lambda M_t + \mu N_t}\right]$$

$$\rightsquigarrow \quad (M_t, N_t) \sim N\left(0, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)$$

$$\rightsquigarrow \quad (M_t - M_s, N_t - N_s) \sim N\left(0, \begin{pmatrix} t-s & 0 \\ 0 & t-s \end{pmatrix}\right)$$

12.7 W: BM. $\tilde{P}$, $X_t = tW_t$.

$$dX_t = t\,dW_t + W_t\,dt \rightarrow \langle X, X\rangle_t = \int_0^t s^2 ds = \frac{t^3}{3}$$

Quadratic variation is invariant under equivalent measure changes because $\langle X, X\rangle_t$ is defined by the limit of

$$\sum_i (X_{t_i} - X_{t_{i-1}})^2.$$

$\therefore$ under $\tilde{\mathbb{P}}$, $\langle X, X \rangle_t = \frac{t^3}{3} \neq t$.

$\therefore$ X is NOT a B.M. under $\tilde{\mathbb{P}}$.

2021 #6

$$\mathbb{E}_s\left[W_t \int_0^t W_r\,dr\right]$$

$$= \mathbb{E}_s\left[W_t \int_0^s W_r\,dr + \underline{W_t \int_s^t W_r\,dr}\right]$$

$$= W_s \cdot \int_0^s W_r\,dr + \mathbb{E}_s\left[W_t \left(\underline{\int_s^t W_r - W_s\,dr}\right) + \underline{W_t \int_s^t W_s\,dr}\right]$$

$$W_s \int_0^t W_s\,ds = W_s \underline{\int_0^t W_r\,dr}$$

$$\int_0^t x\,dx = \int_0^t y\,dy$$

$$= \int_0^t z\,dz$$

2020 #7 α_t , σ_t

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$$

$$\downarrow$$

$$dS_t = r S_t dt + \sigma_t S_t d\tilde{W}_t$$

$$S_T = S_t \cdot e^{\int_t^T (r - \frac{\sigma_s^2}{2}) ds + \int_t^T \sigma_s d\tilde{W}_s}$$

$$V_t = e^{-r(T-t)} \tilde{E}_t [(S_T - K)^+]$$

$$= e^{-r(T-t)} \tilde{E}_t \left[\left(S_t \cdot e^{\int_t^T (r - \frac{\sigma_s^2}{2}) ds + \underline{\int_t^T \sigma_s d\tilde{W}_s}} - K \right)^+ \right]$$

$$\int_t^T \sigma_s d\tilde{W}_s = \int_t^{T_1} \sigma_s d\tilde{W}_s + \int_{T_1}^T \sigma_s d\tilde{W}_s$$

$$= \sigma_1 (\tilde{W}_{T_1} - \tilde{W}_t) + 10\sigma_1 (\tilde{W}_T - \tilde{W}_{T_1})$$

$$\sim N(0, \sigma_1^2 (T_1 - t) + 100\sigma_1^2 (T - T_1))$$

$\rightarrow$ ind of $\mathcal{F}_t$