2019

#6

$$N(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$$N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

2020

#3(b)

If $S(T) > K$, $\quad \tau \to 0$, $\quad x = S(T) > K$

$$d_\pm(\tau, x) = \frac{\log\left(\frac{x}{K}\right)}{\sigma\sqrt{\tau}} + \frac{\left(r \pm \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}$$

as $\tau \to 0$, $\quad d_\pm \longrightarrow \infty$

1. $\langle M, N\rangle = \langle \int_0^\cdot W_s dB_s, \int_0^\cdot B_s dW_s\rangle$

$$= \int_0^t W_s B_s d\langle B, W\rangle_s$$

Two dim Ito

$\rightsquigarrow$ $dM_t^2 N_t^2 =$ martingale $+ (\cdots) dt$

$\Rightarrow$ $\mathbb{E}[M_t^2 N_t^2] = \int_0^t \mathbb{E}[\cdots] dt$

use Ito formula again...

2. Similar to European Call

3. DON'T need to find the formula for Γ_t.

1. $$d\,MN = N\,dM + M\,dN$$

$$M_tN_t = \int_0^t N_s W_s\,dB_s + \int_0^t M_s B_s\,dW_s$$

✓ $$M_t^2N_t^2 = \underbrace{\left(\int_0^t N_s w_s\,dB_s\right)^2}_{} + \left(\int_0^t M_s B_s\,dW_s\right)^2$$

$$+2\left(\int_0^t N_s W_s\,dB_s\right)\left(\int_0^t M_r B_s\,dW_s\right)$$

$$\mathbb{E}[M_t^2N_t^2] = \int_0^t \mathbb{E}[\underbrace{N_s^2 W_s^2}_{=}]\,ds + \int_0^t \mathbb{E}[M_s^2 B_s^2]\,ds$$

$$f(N_s, W_s)$$

$$f(x,y) = x^2y^2$$

2020

#2. $X_t = e^{-2t} W_1(t) + \int_0^t \sigma(s)\, dW_2(s) + \int_0^t b(s)\, ds$

$$dX_t = \underline{-2e^{-2t} W_1(t)\, dt} + e^{-2t} dW_1(t) + \sigma(t) dW_2(t) + \underline{b(t) dt}$$

$$\rightsquigarrow \quad dM_t = \underbrace{e^{-2t} dW_1(t)}_{Y_t} + \underbrace{\sigma(t)\, dW_2(t)}_{Z_t}$$

$$\langle X, X\rangle_t = \langle M, M\rangle_t = \langle Y+Z, Y+Z\rangle_t$$

$$= \langle Y, Y\rangle_t + \langle Z, Z\rangle_t + 2\cdot\langle Y, Z\rangle_t$$

2020

#7 $\quad \sigma(t) = \begin{cases} \sigma_1 & t \le T_1 \\ \sigma_2 & t > T_1 \end{cases}$

$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$ under $\mathbb{P}$

Girsanov

$d S_t = r \cdot S_t dt + \sigma_t S_t d\tilde{W}_t$ under $\tilde{\mathbb{P}}$

$t \le T_1$

$$V_t = e^{-r(T-t)} \tilde{\mathbb{E}}_t\left[(S_T - K)_+\right] \checkmark$$

$$d \log S_t = (r\, dt + \sigma_t d\tilde{W}_t) - \frac{1}{2}\sigma_t^2 dt$$

$$= (r - \tfrac{1}{2}\sigma_t^2) dt + \sigma_t d\tilde{W}_t$$

$$\Rightarrow \quad S_T = S_t \, e^{\int_t^T (r - \frac{1}{2}\sigma_s^2)\, ds + \int_t^T \sigma_s \, d\tilde{W}_s} \quad *$$

$$V_t = e^{-r(T-t)} \underbrace{\tilde{\mathbb{E}}_t \left[\left(S_t \cdot e^{\int_t^T (r - \frac{1}{2}\sigma_s^2) ds + \int_t^T \sigma_s d\tilde{W}_s} - K \right)_+ \right]}$$

$$\left[\begin{aligned} \int_t^T \sigma_s d\tilde{W}_s &= \int_t^{T_1} \sigma_s d\tilde{W}_s + \int_{T_1}^T \sigma_s d\tilde{W}_s \\ &= \sigma_1 (\tilde{W}_{T_1} - \tilde{W}_t) + 10\sigma_1 (\tilde{W}_T - \tilde{W}_{T_1}) \end{aligned} \right]$$

$\Rightarrow \int_t^T \sigma_s d\tilde{W}_s$ is ind of $\mathcal{F}_t$.

$\therefore \quad V_t = e^{-r(T-t)} h(S_t)$ where

$$h(x) = \tilde{\mathbb{E}} \left[\left(x \, e^{\int_t^T (r - \frac{\sigma_s^2}{2}) ds + \underbrace{\int_t^T \sigma_s d\tilde{W}_s}_{\sim N(0, \int_t^T \sigma_s^2 ds)}} - K \right)_+ \right]$$

$= \text{computation} \dots$