1. FCE's → 75% (Pittsburgh)

→ < 75% (NY → )

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2. Final → Practice: → Do problems yourself

↳ a. some problems → standard (Basic tools)

b. little → "more involved problems"

c. Very little → "out of box" thinking

## 12. **Review problems**

*Problem* 12.1. Consider a financial market consisting of a risky asset and a money market account. Suppose the return rate on the money market account is $r$, and the price of the risky asset, denoted by $S$, is a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Here $r, \alpha$ and $\sigma$ are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$\boxed{V_T = \frac{1}{T}\int_0^T S_t\, dt}$$

at maturity $T$. Also compute the trading strategy in the replicating portfolio.

$S = GBM(\alpha, \sigma)$

① Price security: $\quad V_t = \frac{1}{D_t}\tilde{E}_t(D_T V_T)$

$$D_t = \exp\left(-\int_0^t R_s\, ds\right) = e^{-rt}$$

$$\frac{D_T}{D_t} = e^{-r(T-t)}$$

$$(\tau = T - t)$$

$$\Rightarrow V_T = \frac{e^{-r\tau}}{T} \tilde{E}_t \int_0^T S_s \, ds$$

$$= \frac{e^{-r\tau}}{T} \left( \int_0^t S_s \, ds + \int_t^T \tilde{E}_t S_s \, ds \right)$$

$$\underbrace{\tilde{E}_t \int_0^T \sigma_s \, d\tilde{W}_s}_{Mg} = \int_0^t \sigma_s \, d\tilde{W}_s$$

$$\tilde{E}_t \int_0^T b_s \, ds = \int_0^T \tilde{E}_t b_s \, ds$$

$$= \int_0^t b_s \, ds + \int_t^T \tilde{E}_t b_s \, ds$$

Need to compute $\tilde{E}_t S_s$ when $s > t$.

Option 1: Under $\tilde{P}$, $S = GBM(r, \sigma)$

$$\boxed{S_s = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)s + \sigma \tilde{W}_s\right)}$$

$\tilde{W}$ is a BM under $\tilde{P}$.

Indep lemma & compute $\tilde{E}_t(\quad)$.

Better way Option 2:

Under $\tilde{P}$, $e^{-rs} S_s$ is a $\tilde{P}$ mg

$$\therefore \quad \tilde{E}_t(S_s) = e^{rs}\tilde{E}_t(\underbrace{e^{-rs} S_s}_{\tilde{P}\text{ - mg}}) \qquad (s > t)$$

$$= e^{rs}\, e^{-rt} S_t = S_t\, e^{-r(t-s)}$$

Substitute back:

$$V_t = \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + \int_t^T \tilde{E}_t S_s\,ds\right)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + S_t \int_t^T e^{r(s-t)}\,ds\right)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S_s\,ds + \frac{S_t}{r}\left[e^{r(T-t)} - 1\right]\right)$$

Q: Trading Strategy?? $\Delta_t = ??$

$\boxed{\begin{array}{l} \text{Note: If } \underline{\text{Payoff} = g(S_T)} \\ \text{Delta Hedging} \quad \text{Then } \underline{\underline{V_t}} = f(t, S_t) \ \& \ \Delta_t = \partial_x f(t, S_t) \end{array}}$

Let $X_t$ = wealth of R. port. $(\underline{\underline{X_t}} = \underline{\underline{V_t}})$

Know $dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\, dt$

$$= \left( \Delta_t \cancel{rS} - r\cancel{\Delta_t S} + rX_t \right) dt + \underline{\Delta_t} \sigma S_t \, \underline{\underline{d\tilde{W}}}$$

$$\boxed{dS = rS\,dt + \underline{\sigma} S\, d\tilde{W}}$$

Use formula for V & find coeff of $d\tilde{W}$

$$V_t \overset{Le}{=} \frac{e^{-r\tau}}{\underline{\underline{T}}} \left( \underline{\underline{\int_0^t S_s \, ds}} + \frac{S_t}{r} \left[ e^{r(T-t)} - 1 \right] \right)$$

Want this

$$\Rightarrow dV_t = \left( \text{uuu} \right) dt + \left( \quad \right) d\tilde{W}$$

$$dV_t = d\left(e^{(\,)} \int_0^t \; ds\right) + d\left[\frac{S_t}{r}\left(e^{r(T-t)} - 1\right) \cdot \frac{e^{-r\tau}}{T}\right]$$

$$= \left(m\,dt + S_t\, d(\,) + \frac{1 - e^{-r\tau}}{rT}\, dS_t + d[\,,\,]\right.$$

$$= (\text{smthg})\,dt + \boxed{\frac{1 - e^{-r\tau}}{rT}\,\sigma S_t\, d\widetilde{W}}$$

equate this to $\Delta_t \sigma S_t$.

$$\Rightarrow \Delta_t = \boxed{\frac{1 - e^{-r\tau}}{r\tau}}$$

*Problem* 12.2. Let $X \sim N(0,1)$, and $a, \alpha, \beta \in \mathbb{R}$. Define a new measure $\tilde{\boldsymbol{P}}$ by

$$\boxed{d\tilde{\boldsymbol{P}} = \exp(\alpha X + \beta)\, d\boldsymbol{P}\,.}$$

Find $\alpha, \beta$ such that $X + a \sim N(0,1)$ under $\tilde{\boldsymbol{P}}$.

Sol: Compute $\tilde{E}\left(e^{\lambda(X+a)}\right) \overset{\text{Want}}{=} e^{\lambda^2/2}$

$$\tilde{E} e^{\lambda(X+a)} = \int_\Omega e^{\lambda(X+a)}\, d\tilde{P} = \int_\Omega e^{\lambda(X+a)} e^{(\alpha X+\beta)}\, dP$$

$$= E\left(e^{\lambda(X+a)} e^{(\alpha X+\beta)}\right)$$

$$= E\, e^{(\lambda+\alpha)X + \lambda a + \beta}$$

$$= e^{\lambda a + \beta + (\lambda+\alpha)^2/2} \quad \underline{\underline{\text{Want}}} \quad e^{\lambda^2/2}.$$

Equate $\boxed{\lambda a + \beta + \frac{(\lambda+\alpha)^2}{2} = \frac{\lambda^2}{2} \text{ \& solve.}}$

*Problem* 12.3. Let $f$ be a deterministic function, and define

$$X_t \stackrel{\text{def}}{=} \int_0^t f(s) W_s \, ds\,.$$

Find the distribution of $X$.

Guess: $\int_0^t f(s) W_s \, ds \sim N$

Reason: $\int_0^t f(s) W_s \, ds = \lim_{\|P\| \to 0} \underbrace{\sum \underbrace{f(t_i)}_{\text{not random}} W_{t_i} \underbrace{(t_{i+1} - t_i)}_{\text{not random}}}_{\text{linear comb of a Joint normal}}$

Normal !!

Need to find $EX_t$ & $\operatorname{Var} X_t$

① $EX_t = E\int_0^t f(s) W_s \, ds = \int_0^t f(s)\, EW_t \, ds = 0$

② $EX_t^2 = E\left(\int_0^t f(s) W_s \, \underline{\underline{ds}}\right)^2 =$

$= E\left(\int_0^t f(s) W_s \, ds\right)\left(\int_0^t f(\cancel{s}\,r) W_{\cancel{s}\,r} \, d\cancel{s}\,r\right)$

Do Isom:

$E\left(\int_0^t \tau_s \, dW_s\right)^2 = E\int_0^t \tau_s^2 \, ds$

$$= E \int_{s=0}^{t} \int_{r=0}^{t} f(s)\, f(r)\, W_s W_r \, dr \, ds$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} f(s)\, f(r) \underbrace{E\left(W_s W_r\right)}_{s \wedge r \ (s \min r)} dr \, ds$$

& compute.

*Problem* 12.4. Let $x_0, \mu, \theta, \sigma \in \mathbb{R}$, and suppose $X$ is an Itô process that satisfies

$$dX(t) = \theta(\mu - X_t)\, dt + \sigma\, dW_t\,,$$

(O. U. Process)

with $X_0 = x_0$.

(a) Find functions $f = f(t)$ and $g = g(s,t)$ such that

$$\boxed{X(t) = f(t) + \int_0^t g(s,t)\, dW_s\,.}$$

The functions $f, g$ may depend on the parameters $x_0, \theta, \mu$ and $\sigma$, but should not depend on $X$.

(b) Compute $\boldsymbol{E} X_t$ and $\operatorname{cov}(X_s, X_t)$ explicitly.

(a) Compute $d\left(e^{\theta t} X_t\right) = e^{\theta t}\, dX + X_t\, \theta e^{\theta t}\, dt + \underbrace{d[e^{\theta t}, X]_t}_{0}$

$$= e^{\theta t}\left(\theta(\mu - X)\,dt + \sigma\, dW\right) + X\,\theta\, e^{\theta t}\, dt$$

$$= e^{\theta t}\theta\mu\, dt + e^{\theta t}\sigma\, dW_t$$

$$\Rightarrow \underline{e^{\theta t} X_t} - X_0 = \mu \int_0^t \theta e^{\theta s}\, ds + \int_0^t \sigma e^{\theta s}\, dW_s$$

$$= \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s}\, dW_s$$

$$\Rightarrow \underline{X_t} = \underbrace{e^{-\theta t} X_0 + \mu\left(1 - e^{-\theta t}\right)}_{EX_t} + \sigma \underbrace{e^{-\theta t} \int_0^t e^{\theta s}\, dW_s}_{E=0}$$

(b) Compute $\mathrm{Cov}(X_s, X_t)$

$$\mathrm{Cov}(X_s, X_t) = E\left(X_s - EX_s\right)\left(X_t - EX_t\right)$$

$$= E\left(\sigma e^{-\theta s}\int_0^s e^{\theta r}\,dW_r\right)\left(\sigma e^{-\theta t}\int_0^t e^{\theta r}\,dW_r\right)$$

$$= \sigma^2 e^{-\theta(s+t)} E\left(\int_0^s e^{\theta r}\,dW_r \int_0^t e^{\theta r}\,dW_r\right)$$

$$= \sigma^2 e^{-\theta(s+t)} E\left(\int_0^s e^{\theta r}\,dW_r \left(\int_0^s e^{\theta r}\,dW_r + \int_s^t e^{\theta r}\,dW_r\right)\right)$$

$$= \sigma^2 e^{-\theta(s+t)} \left[ E\left( \int_0^s e^{\theta r}\, dW_r \right)^2 + E\left[ \underbrace{\int_0^s e^{\theta r}\, dW_r}_{\mathcal{F}_s \text{ meas}} \underbrace{\int_s^t \underbrace{e^{\theta r}}_{\text{not random}} \underbrace{dW_r}_{\mathcal{F}_s\text{-indep}}}_{\text{Indep of } \mathcal{F}_s} \right] \right]$$

$$= \sigma^2 e^{-\theta(s+t)} \left[ \cancel{E} \int_0^s e^{2\theta r}\, dr + 0 \right]$$

& compute.

*Problem* 12.5. Let $W$ be a Brownian motion, and define

$$B_t = \int_0^t \operatorname{sign}(W_s)\, dW_s\,.$$

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

(a) Show that $B$ is a Brownian motion.

(b) Is there an adapted process $\sigma$ such that

$$W_t = \int_0^t \sigma_s\, dB_s\,?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation $[B, W]$.

(d) Are $B$ and $W$ uncorrelated? Are they independent? Justify.

(a) : Levy's Criterion : Need ① $X$ is a cts Martingale

② $[X, X]_t = t$

① $X$ is a cts mg since $dX_t = (\ )\, dW_t$ (Ito int is a mg)

(2) $d[X,X]_t = \mathrm{sign}(W_t)^2\, dt = 1\, dt$

$\Rightarrow$ (a).

(b) $dB_s = \mathrm{sign}(W_s)\, dW_s \Rightarrow dW_s = \frac{1}{\mathrm{sign}(W_s)} dB_s = \mathrm{sign}(W_s)\, dB_s$

$$\Rightarrow W_t = \int_0^t \mathrm{sign}(W_s)\, dB_s$$

(c) Compute $[B, W]$:

$$dB_t = \text{sign}(W_t)\, dW_t$$
$$dW_t = 1 \quad dW_t$$

$$d[B, W] = \text{Sign}(W_t) \cdot 1 \cdot d[W_t, W_t]$$
$$= \text{Sign}(W_t)\, dt$$

(d). Are $B$ & $W$ uncor?

$$\left.\begin{aligned} EB_t &= 0 \\ EW_t &= 0 \end{aligned}\right\} \quad \text{Compute } E(B_t W_t)$$

$u$

$$d(B_t W_t) = B_t\, dW_t + W_t\, dB_t + d[B, W]_t$$

$$= B_t\, dW_t + W_t \operatorname{sign}(W_t)\, dW_t + \operatorname{sign}(W_t)\, dt$$

$$\Rightarrow B_t W_t - \underbrace{B_0 W_0}_{0} = \int_0^t \left(B_s + W_s \operatorname{sign}(W_s)\right) dW_s + \int_0^t \operatorname{sign}(W_s)\, ds.$$

$$\Rightarrow E(B_t W_t) = E \int_0^t (\quad)\, dW_s + E \int_0^t \operatorname{sign}(W_s)\, ds$$

$$= 0 + \int_0^t \underbrace{E\,\mathrm{sign}(W_s)}_{0}\, ds$$

$\Rightarrow E(B_t W_t) = 0 \Rightarrow B_t$ & $W_t$ are uncor!!

Q: Are $B_t$ & $W_t$ indep?

$B_t$ & $W_t$ are Norml & uncor

$\not\Rightarrow$ Indep (Not Jointly normal).

B & W are NOT indep

because If B & W were indep then $[B, W] = 0$

But $d[B, W]_t = \operatorname{Sign}(W_t)\, dt \neq 0$.

*Problem* 12.6. Suppose $\sigma, \tau, \rho$ are three deterministic functions and $M$ and $N$ are two continuous martingales with respect to a common filtration $\{\mathcal{F}_t\}$ such that $M_0 = N_0 = 0$, and

$$d[M, M]_t = \sigma_t \, dt \,, \quad d[N, N]_t = \tau_t \, dt \,, \qquad \text{and} \qquad d[M, N]_t = \rho_t \, dt \,.$$

(a) Compute the joint moment generating function $\boldsymbol{E} \exp(\lambda M(t) + \mu N(t))$.
(b) *(Lévy's criterion)* If $\sigma = \tau = 1$ and $\rho = 0$, show that $(M, N)$ is a two dimensional Brownian motion.

*Problem* 12.7. Let $W$ be a Brownian motion. Does there exist an equivalent measure $\tilde{\boldsymbol{P}}$ under which the process $tW_t$ is a Brownian motion? Prove it.

*Problem* 12.8. Let $\theta \in \mathbb{R}$ and define

$$Z_t = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big)\,.$$

Given $0 \leqslant s < t$, and a function $f$, find a function such that

$$\boldsymbol{E}_s f(Z_t) = g(Z_s)\,.$$

Your formula for the function $g$ can involve $f$, $s$, $t$ and integrals, but not the process $Z$ or expectations.

*Problem* 12.9. Consider the $N$ period Binomial model with $N = 5$, and parameters $0 < d < 1 + r < u$. At maturity $N = 5$, a security pays \$1 if $S_5 > (1 + r)S_4$, and 0 otherwise. Find the arbitrage free price and trading strategy trading at time 0.