## 12. Review problems

Problem 12.1. Consider a financial market consisting of a risky asset and a money market account. Suppose the return rate on the money market account is r and the price of the risky asset, denoted by S, is a geometric Brownian motion with mean return rate  $\alpha$  and volatility  $\sigma$ . Here  $r, \alpha$  and  $\sigma$  are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$V_{\underline{T}} = \frac{1}{T} \int_0^T S_t \, dt$$

at maturity T. Also compute the trading strategy in the replicating portfolio.

$$S = GEU(x,r)$$
  

$$OP_{vie} \quad seawly: \quad V_{t} = \frac{1}{P_{t}} E_{t}(D_{T}V_{t})$$
  

$$P_{t} = eap(-\int_{0}^{t} R_{s} ds) = e^{-rt}$$

 $\frac{D}{T} = e^{-r(T-t)}$  $\Rightarrow$  V<sub>T</sub> =  $e^{TE}E_{t}$   $\int S_{s} dS$  $= \frac{e^{T}}{T} \left( \int_{0}^{t} S_{s} ds + \int_{0}^{t} E_{t} S_{s} ds \right)$ 

(T = T - t) $\tilde{E}_{t}$   $\int \sigma_{s} dW_{s} = \int \sigma_{s} dW_{s}$  $\tilde{E}_{t} \int_{S} t_{s} ds = \int_{b} \tilde{E}_{b} ds$ = jbsds+ ( JE<sub>t</sub>b<sub>s</sub>ds

Noo b compute  $\tilde{E}_t S_s$  when s > t. Option 1 8 Under P,  $S = GBM(r, \tau)$  $S_{S} = S_{S} eab((a - \tau_{Z}^{2})S + \tau W_{S})$   $\widetilde{W} is a BM undu \widetilde{P}.$ Indep deux 2 compte Et. ().

Boller way Ophen 2: Under P, end Ss is a P ma  $\stackrel{\circ}{\bullet} \stackrel{\mathcal{V}}{\mathsf{E}}_{\mathsf{f}}(\mathsf{S}_{\mathsf{S}}) = \overset{\mathsf{re}}{\mathsf{e}} \stackrel{\mathcal{V}}{\mathsf{E}}_{\mathsf{f}}(\overset{\mathsf{re}}{\mathsf{e}} \overset{\mathsf{S}}{\mathsf{S}}) \quad (\mathsf{s} > \mathsf{t})$ P-ma  $= e^{rs} e^{-rt} S_{t} = S_{t} e^{-r(t-s)}$ 

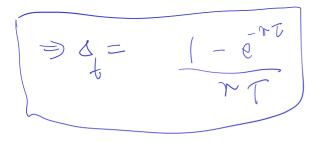
Slibstyne burk:

 $V_{t} = \frac{e^{T}}{T} \left( \int_{0}^{t} S_{s} ds + \int_{0}^{t} E_{t} S_{s} ds \right)$  $= \frac{e^{TT}}{T} \left( \int_{0}^{t} S \, dS + S_{t} \int_{0}^{T} e^{r(s-t)} \, dS \right)$  $= \frac{e^{-\lambda t}}{T} \left( \int \frac{t}{S} dS + \frac{S}{T} \left[ e^{\lambda t} \left[ e^{\lambda t} - 1 \right] \right] \right) dx$ 

 $Q^{\circ}$  Trading Strategy ??  $\Delta_1 = ??$ Note: If Poyoff =  $g(S_T)$ Delta Hodging Then  $V_t = f(t, S_t) & a_t = 2 f(t, S_t)$ hat  $X = weath of R \cdot fast. (X = V_{t})$ Know  $dX_t = \Delta_t dS_t + m(X_t - A_tS_t) dt$ 

 $= \begin{pmatrix} 4\pi s - \pi 4s \\ +\pi X_t \end{pmatrix} dt + 4t S_t dW$ Use feula for V & fand coff of div  $V_{t} = \frac{e^{-ME}}{T} \int \frac{f}{s} ds + \frac{s}{T} \left[ e^{r(T-t)} - 1 \right]$  Wat dwa,  $\Rightarrow dV_{t} = (MM) dt + () dW$ 

 $dV_{t} = d(e^{C}) \int_{0}^{t} ds + d\left(\sum_{T=0}^{S_{t}} (e^{T-t}) - 1\right) \cdot e^{-TT}$  $= \left( \frac{1}{1} \frac{1}{1} + \frac{1}{2} \frac{1}{2} \right) + \left( \frac{1}{1} - \frac{1}{2} + \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} \right)$  $+\underbrace{1-e^{-rT}}_{rT}+S_{t}W$ = (south ) of t equile this to At, TSL.



 $= E e^{(\lambda + \alpha)X + \lambda \alpha + \beta}$ Equale  $\lambda_{n+p+\tau} \left(\frac{\lambda_{t\alpha}}{2}\right)^2 = \frac{\lambda^2}{2} \& c_{\alpha} v_{e}.$ 

Problem 12.3. Let f be a deterministic function, and define

North find EX, & Var X,  $(C) \in X_{t_s} = E \int_{a}^{b} f(s) W_s ds =$  $\int ds = O$  $(2) E X_{\pm}^{2} = E\left(\int_{0}^{t} f(s) W_{s} ds\right)^{2} = \begin{bmatrix} H_{0} & \text{Isom} \\ F_{s} dW_{s} \end{bmatrix} = E\left(\int_{0}^{t} F_{s} dW_{s}\right)^{2} = E\left(\int_{0}^{t} F_{$  $= \mathbb{E}\left(\int_{-\infty}^{\infty} f(s) W_{s} ds\right)\left(\int_{-\infty}^{\infty} f(s) W_{s} ds\right)$ 

$$= E \int \int f(s) f(r) W_s W_r dr ds$$
  

$$= \int \int \int f(s) f(r) E(W_s W_r) dr dc$$
  

$$= \int \int f(s) f(r) E(W_s W_r) dr dc$$
  

$$= \int \int r=0$$
  
SAT (c min r) & compute.

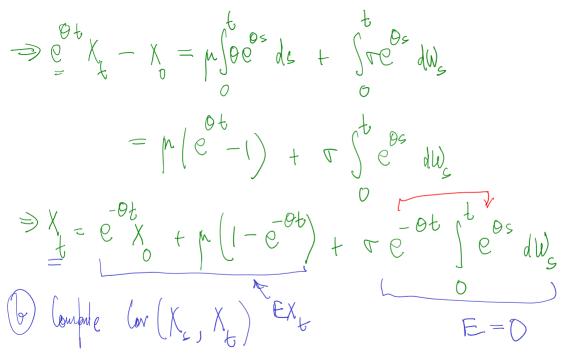
Problem 12.4. Let  $x_0, \mu, \theta, \sigma \in \mathbb{R}$ , and suppose <u>X</u> is an Itô process that satisfies

$$\underline{dX(t)} = \underbrace{\theta(\mu - X_t)}_{t} dt + \sigma dW_t, \qquad (\bigcirc \bigcup W_t, \qquad (\bigcirc U) \\ \underbrace{Waeets}_{t} \end{pmatrix}$$

$$dX(t) = \underbrace{\theta}(\mu - X_t) dt + \sigma dW_t,$$
  
with  $X_0 = x_0$ .  
(a) Find functions  $f = f(t)$  and  $g = g(s, t)$  such that  
$$X(t) = f(t) + \int_0^t g(s, t) dW_s.$$

The functions f, g may depend on the parameters  $x_0, \theta, \mu$  and  $\sigma$ , but should not depend on X. (b) Compute  $\boldsymbol{E}X_t$  and  $\operatorname{cov}(X_s, X_t)$  explicitly.

(a) Compute 
$$d(e^{\Theta t}X_t) = e^{\Theta t}dx + X_t \Theta e^{\Theta t}dt + d[e^{\Theta t}X_t]_t$$
  
 $= e^{\Theta t}(\Theta(\mu - X)dt + \tau dW) + X \Theta e^{\Theta t}dt$   
 $= e^{\Theta t}\Theta \mu dt + e^{\Theta t}\tau dW_t$ 



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 $C_{w}(X_{s},X_{t}) = E(X_{s}-EX_{t})(X_{t}-EX_{t})$ 

 $= E\left(re^{-\Theta s}\int_{c}^{c}e^{-AW}r\right)\left(re^{-\Theta t}\int_{c}^{t}e^{-AW}r\right)$  $= r^{2} e^{-\Theta(s+t)} E\left(\int_{0}^{s} e^{\Theta r} dW_{r} \int_{0}^{t} e^{\Theta r} dW_{r}\right)$  $= \nabla^{2} e^{-\theta(c_{t}t_{t})} E\left(\int_{0}^{s} e^{\theta r} dW_{r} \left(\int_{0}^{s} e^{\theta r} dW_{r} + \int_{s}^{t} e^{\theta r} dW_{r}\right)\right)$ 

 $= r^{2} e^{-\Theta(s+b)} \left[ E\left(\int_{0}^{s} e^{\Theta r} dw_{r}\right) + E\left(\int_{0}^{s} e^{\Theta r} dw_{r}\right) \int_{0}^{t} e^{\Theta r} dw_{r}\right] \right]$  $= \nabla^2$ e<sup>207</sup>dr +0 Æ l'compute.

Problem 12.5. Let W be a Brownian motion, and define

$$\underline{B}_t = \int_0^t \operatorname{sign}(W_s) \, dW_s \, .$$

 $Sign(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x \ge 0 \end{cases}$ 

(a) Show that B is a Brownian motion.

(b) Is there an adapted process  $\sigma$  such that

$$W_t = \int_0^t \sigma_s \, dB_s \, ?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation [B, W].

(d) Are B and W uncorrelated? Are they independent? Justify.

D: Long's Childrin: Now DX is a 
$$dS$$
 Mantigale  
 $(\overline{Z} [X, X]_{5} = t$   
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 $(\exists d[X,X]_{\pm} = sign(W_{\pm})^{2} dt = 1 dt$ 

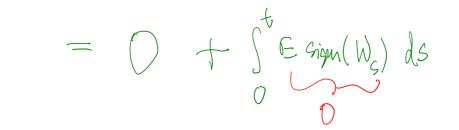
 $\rightarrow$  (D)

 $(f) dB_{S} = sign(W_{S}) dW_{S} \implies dW_{S} = \frac{1}{sign(W_{S})} dB_{S} = sign(W_{S}) dB_{S}$  $\gg W_{t} = \int Sign(W_{s}) dB_{s}$ 

$$\begin{array}{l} \hline (c) \quad (c) \quad (c) \quad (d) \quad (d$$

(d). Ane B&W uner:  $EB_{t} = 0$  { compute  $E(B_{t}W_{t})$  $EW_{t} = 0$  } IK

 $d(B, W_{t}) = B_{t}dW_{t} + W_{t}dB_{t} + d[B, W]_{t}$  $= B_{f} dW_{f} + W_{f} sign(W_{f}) dW_{f} + Sign(W_{f}) dt$  $\Rightarrow B_{1}W_{1} - B_{0}W_{0} = \int (B_{2} + W_{c} H_{y}(W_{c})) dW_{c} + \int S_{1}y_{1}(W_{c}) dS.$  $\Rightarrow E(B_{t}W_{t}) = E \int () dW_{t} + E \int Sp_{t}(W_{t}) ds$ 



 $\Rightarrow E(B_{t}W_{t}) = 0 \Rightarrow B_{t} U_{t} w_{t} w_{t}$ 

Q: Ane B & W indep? Bz 2W are Nound & uncon Dindle (Not Jointly nound).

B& W ane NOT indép because IL B&W were indép Hur [B,W]=0 But  $d[B,W]_{t} = Sign(W_{t}) dt \neq 0$ ,

Problem 12.6. Suppose  $\sigma, \tau, \rho$  are three deterministic functions and M and N are two continuous martingales with respect to a common filtration  $\{\mathcal{F}_t\}$  such that  $M_0 = N_0 = 0$ , and

$$d[M,M]_t = \sigma_t \, dt \,, \quad d[N,N]_t = \tau_t \, dt \,, \qquad \text{and} \qquad d[M,N]_t = \rho_t \, dt \,.$$

(a) Compute the joint moment generating function  $\boldsymbol{E} \exp(\lambda M(t) + \mu N(t))$ .

(b) (Lévy's criterion) If  $\sigma = \tau = 1$  and  $\rho = 0$ , show that (M, N) is a two dimensional Brownian motion.

Problem 12.7. Let W be a Brownian motion. Does there exist an equivalent measure  $\tilde{P}$  under which the process  $tW_t$  is a Brownian motion? Prove it.

Problem 12.8. Let  $\theta \in \mathbb{R}$  and define

$$Z_t = \exp\left(\theta W_t - \frac{\theta^2 t}{2}\right).$$

Given  $0 \leq s < t$ , and a function f, find a function such that

$$\boldsymbol{E}_s f(\boldsymbol{Z}_t) = g(\boldsymbol{Z}_s) \,.$$

Your formula for the function g can involve f, s, t and integrals, but not the process Z or expectations.

Problem 12.9. Consider the N period Binomial model with N = 5, and parameters 0 < d < 1 + r < u. At maturity N = 5, a security pays \$1 if  $S_5 > (1 + r)S_4$ , and 0 otherwise. Find the arbitrage free price and trading strategy trading at time 0.