Thm (Girsanov) W_t : $\mathbb{P}$-B.M. b: Conti adapted.

$$Z_t = e^{-\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds}$$

$$d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P}$$

If Z_t is a mart, then $\tilde{W}_t = W_t + \int_0^t b_s ds$ is a $\tilde{\mathbb{P}}$-BM

Q. How can we ~~guarantee~~ that Z_t is a mart?

$dZ_t = \cdots dW_t$

A. sufficient conditions: $\mathbb{E}[e^{\frac{1}{2}\int_0^t b_s ds}] < \infty$ (Novikov) or $\mathbb{E}[Z_T] = 1$ or ......

Rmk Meaning of $d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P}$

- $\tilde{\mathbb{P}}(A) = \mathbb{E}[\mathbb{1}_A \cdot Z_T]$
- $\tilde{\mathbb{E}}[X] = \mathbb{E}[X \cdot Z_T]$ ✓
- $\tilde{\mathbb{E}}_t[X] = \mathbb{E}_t[X \cdot Z_T]$? NO!

Lem $\tilde{E}_s[X_t] = \dfrac{E_s[X_t Z_t]}{Z_s}$ $\quad t \geq s$

COR M is $\tilde{P}$ - mart $\iff$ $Z \cdot M$ is P - mart

EX (HW4 #2) Prove X is a mart

$$X_t = \left(W_t + \int_0^t b_s\,ds\right) e^{-\int_0^t b_s\,dW_s - \frac{1}{2}\int_0^t b_s^2\,ds}$$

$$= \tilde{W}_t \cdot Z_t$$

By COR, X is P-mart $\iff$ $\tilde{W}$ is $\tilde{P}$ - mart

EX $E\left[e^{W_T}\int_0^T W_s^2\,ds\right] = ?$

$$E\left[e^{W_T}\int_0^T W_s^2\,ds\right] = E\left[\underline{e^{W_T - \frac{1}{2}T}}\int_0^T W_s^2\,ds\right]e^{T/2} = E\left[Z_T\int_0^T W_s^2\,ds\right]e^{T/2}$$

(when $b = -1$, $Z_t = e^{W_t - \frac{1}{2}t}$, $\tilde{W}_t = W_t - t$ is $\tilde{P}$ - BM.)

$$= \tilde{\mathbb{E}}\left[\int_0^T W_s^2 ds\right] e^{T/2} = \tilde{\mathbb{E}}\left[\int_0^T (\tilde{W}_s + s)^2 ds\right] e^{T/2}$$

$$= \int_0^T \tilde{\mathbb{E}}\left[(\tilde{W}_s + s)^2\right] ds \, e^{T/2}$$

$$= \int_0^T (s + s^2) ds \cdot e^{T/2} = \left(\frac{T}{2} + \frac{T^3}{3}\right) e^{T/2}.$$

<u>Ex</u> $\mathbb{E}_t\left[e^{W_T} \int_0^T W_s^2 ds\right] = ?$

$$\mathbb{E}_t\left[e^{W_T} \int_0^T W_s^2 ds\right] = \mathbb{E}_t\left[Z_T \cdot \int_0^T W_s^2 ds\right] \cdot e^{T/2}$$

$$= Z_t \cdot \tilde{\mathbb{E}}_t\left[\int_0^T W_s^2 ds\right] \cdot e^{T/2} \quad \text{(by cem)}.$$

$$= Z_t \cdot \tilde{\mathbb{E}}_t\left[\int_0^T (\tilde{W}_s + s)^2 ds\right] \cdot e^{T/2}$$

$$= Z_t \cdot \int_0^T \tilde{\mathbb{E}}_t\left[(\tilde{W}_s + s)\right]^2 ds \cdot e^{T/2}$$

$$= Z_t \cdot \left(\int_0^t (\tilde{W}_s + s)^2 ds + \underbrace{\int_t^T \tilde{\mathbb{E}}_t\left[(\tilde{W}_s + s)^2\right] ds}_{\text{*}}\right) e^{T/2}$$

$$(\star) = \int_t^T \tilde{\mathbb{E}}_t\left[(\tilde{W}_s + s)^2\right] ds$$

$$= \int_t^T \tilde{\mathbb{E}}_t[\tilde{W}_s^2] + 2s\, \tilde{\mathbb{E}}_t[\tilde{W}_s] + s^2 \, ds$$

$$= \int_t^T (\tilde{W}_t^2 - t + s) + 2s \cdot \tilde{W}_t + s^2 \, ds$$

$$= (\tilde{W}_t^2 - t)(T-t) + \frac{T^2 - t^2}{2} + \tilde{W}_t (T^2 - t^2) + \frac{T^3 - t^3}{3}.$$

$$\therefore \mathbb{E}_t\left[e^{W_T} \int_0^T W_s^2 \, ds\right]$$

$$= \left(e^{W_t - \frac{1}{2}t}\right)\left(\int_0^t (\tilde{W}_s + s)^2 ds + (\star)\right) \cdot e^{T/2}.$$

<u>EX</u> $dX_t = \alpha_t\, dt + \beta_t\, dW_t$ (Ito process) $\overset{\text{Girsanov}}{\rightsquigarrow}$ $dX_t = \alpha_t'\, dt + \beta_t\, d\tilde{W}_t$

$dX_t = t \sin t \, dt + t\, dW_t$ under $\mathbb{P}$.

$\left(Z_t = e^{-\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds}, \quad d\tilde{\mathbb{P}} = Z_T\, d\mathbb{P}, \quad d\tilde{W}_t = dW_t + b_t dt \right.$ is a $\tilde{\mathbb{P}}$-BM. $\Big)$

under $\tilde{\mathbb{P}}$, $dX_t = t\sin t\,dt + t\cdot(d\tilde{W}_t - b_t\,dt)$

$$= (t\cdot\sin t - t b_t)\,dt + t\cdot d\tilde{W}_t$$

If we choose $b_t = \sin t$, $dX_t = t\cdot d\tilde{W}_t$ under $\tilde{\mathbb{P}}$.

In particular, X is a martingale under $\tilde{\mathbb{P}}$.

Ex $dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$ under $\mathbb{P}$.

(under $\tilde{\mathbb{P}}$, $d\tilde{W}_t = dW_t + b_t\,dt$ is a B.M.)

$$dS_t = \alpha S_t\,dt + \sigma S_t(d\tilde{W}_t - b_t\,dt) \text{ under } \tilde{\mathbb{P}}$$

$$= (\alpha - \sigma b_t) S_t\,dt + \sigma S_t\,d\tilde{W}_t$$

If we choose $b_t = \dfrac{\alpha - r}{\sigma}$

$$= r S_t\,dt + \sigma S_t\,d\tilde{W}_t$$

✓

Black-Scholes Model

risk-free rate : R_t

Discount factor : $D_t = e^{-\int_t^T R_s ds}$

Stock : $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

Def (RN) $\tilde{\mathbb{P}}$ is a RN if

① $\tilde{\mathbb{P}}$ is equiv to $\mathbb{P}$ ② $D_t S_t$ is $\tilde{\mathbb{P}}$-mart.

Thm 10.4 There is a unique RN $\tilde{\mathbb{P}}$.

$$d\tilde{\mathbb{P}} = Z_T d\mathbb{P} \quad \text{where} \quad Z_T = e^{-\int_0^T b_s dW_s - \frac{1}{2}\int_0^T b_s^2 ds},$$

$$b_s = \frac{\alpha_s - R_s}{\sigma_s}$$

Rmk under $\tilde{\mathbb{P}}$, $dS_t = \alpha_t S_t dt + \sigma_t S_t \left(d\tilde{W}_t - \left(\frac{\alpha_t - R_t}{\sigma_t}\right) dt \right)$

$$= R_t S_t dt + \sigma_t S_t d\tilde{W}_t$$

Thm 10.5 (pricing formula) A security pays V_T at time T,

AFP of this security at time t, $V_t = \frac{1}{D_t} \tilde{\mathbb{E}}_t [D_T V_T]$

Black-scholes Formula

$R_t = r$, $\alpha_t = \alpha$, $\sigma_t = \sigma$

Thm 11.1 $V_T = g(S_T)$.

AFP of this security at time t, $V_t = f(t, S_t)$ where

$$f(t,x) = e^{-r\tau} \int_{-\infty}^{\infty} g\left(x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y}\right) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}, \quad \tau = T-t.$$

pf $V_t = e^{-r(T-t)} \tilde{\mathbb{E}}_t [g(S_T)]$

under $\tilde{\mathbb{P}}$, $dS_t = r \cdot S_t dt + \sigma S_t d\tilde{W}_t$

$$d \log S_t = \frac{1}{S_t} dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right) d\langle S, S\rangle_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma d\tilde{W}_t$$

$$\Rightarrow \log(S_t) - \log(S_0) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t$$

$$\Rightarrow S_t = S_0 \cdot e^{\left(r-\frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t}$$

$$\Rightarrow S_T = S_t \cdot e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)}$$

$$\therefore V_t = e^{-r(T-t)} \tilde{\mathbb{E}}_t\left[g\left(S_t \cdot e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)}\right)\right]$$

$$= e^{-r(T-t)} h(S_t)$$

where $h(x) = \tilde{\mathbb{E}}\left[g\left(x\, e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \underbrace{\sigma(\tilde{W}_T - \tilde{W}_t)}_{= \sigma \cdot \sqrt{T-t} \cdot Z,\ Z \sim N(0,1)}}\right)\right]$

$$= \int_{-\infty}^{\infty} g\left(x \cdot e^{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\cdot y}\right) \cdot \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}.$$

2019 Final, W, B Ind B.M.

$X_t = W_t^2 + W_t B_t + t^2$.

$\langle X, X \rangle_t = ?$

$\langle X, W \rangle_t = ?$

$X_t = g(t, W_t, B_t)$ where $g(t,x,y) = x^2 + xy + t^2$

$\partial_{xx} g = 2$

$\partial_{yy} g = 0$

$\partial_{xy} g = 1$

$$dX_t = 2t\,dt + (2W_t + B_t)\,dW_t + W_t\,dB_t$$

$$+ \frac{1}{2}\cdot 2 \cdot \underbrace{d\langle W,W\rangle_t}_{=dt} + \cancel{\frac{1}{2}\cdot 0 \cdot d\langle B,B\rangle_t}$$

$$+ 1 \cdot \cancel{d\langle W,B\rangle_t} \; 0$$

$$= \underline{(2t+1)\,dt} + (2W_t+B_t)\,dW_t + W_t\,dB_t$$

$$\langle X,X\rangle_t = \Big\langle \overset{Y_t}{\underline{\int_0^t (2W_s+B_s)\,dW_s}} + \overset{Z_t}{\underline{\int_0^t W_s\,dB_s}},\ \int_0^t (2W_s+B_s)\,dW_s + \int_0^t W_s\,dB_s \Big\rangle$$

$$= \langle Y,Y\rangle_t + 2\langle Y,Z\rangle_t + \langle Z,Z\rangle_t = \langle Y,Y\rangle_t + \langle Z,Z\rangle_t$$

This is because $\langle Y,Z\rangle_t = \int_0^t (2W_s+B_s)\cdot W_s\, d\langle W,B\rangle_s = 0$.

Ex $dS_t = \alpha_t S_t \, dt + t S_t \, dW_t$, $r > 0$

under $\tilde{\mathbb{P}}$, $dS_t = r S_t \, dt + t \cdot S_t \, d\tilde{W}_t$

price Call option : $(S_T - K)_+$

$$V_t = e^{-r(T-t)} \tilde{\mathbb{E}}_t \left[(S_T - K)_+ \right]$$

$$d \log(S_t) = \frac{1}{S_t} \cdot dS_t - \frac{1}{2 S_t^2} d\langle S, S \rangle_t$$

$$= (r \, dt + t \, d\tilde{W}_t) - \frac{1}{2 S_t^2} \cdot t^2 S_t^2 \, dt$$

$$= \left(r - \frac{t^2}{2}\right) dt + t \, d\tilde{W}_t$$

$$\Rightarrow S_t = S_0 \cdot e^{\int_0^t \left(r - \frac{s^2}{2}\right) ds + \int_0^t s \, d\tilde{W}_s}$$

$$\Rightarrow S_T = S_t \cdot e^{\int_t^T \left(r - \frac{s^2}{2}\right) ds + \int_t^T s \, d\tilde{W}_s}$$

$$V_t = e^{-r(T-t)}\, \tilde{\mathbb{E}}_t\left[\left(S_t \cdot e^{\int_t^T (r-\frac{\sigma^2}{2})ds + \int_t^T \sigma d\tilde{W}_s} - K\right)_+\right]$$

- $\int_t^T \sigma d\tilde{W}_s = \lim \sum_i \sigma_{t_i}\left(\tilde{W}_{s_{t_{i+1}}} - \tilde{W}_{s_{t_i}}\right)$ Ind of $\mathcal{F}_t$
- $\int_t^T \sigma d\tilde{W}_s \sim N\left(0, \int_t^T \sigma^2 ds\right)$

$$= e^{-r(T-t)} \cdot h(S_t) \quad \text{where}$$

$$h(x) = \tilde{\mathbb{E}}\left[\left(x \cdot e^{\int_t^T (r-\frac{\sigma^2}{2})ds + \int_t^T \sigma d\tilde{W}_s} - K\right)_+\right]$$

$$= \tilde{\mathbb{E}}\left[\left(x \cdot e^{\int_t^T (r-\frac{\sigma^2}{2})ds + \sqrt{\int_t^T \sigma^2 ds}\cdot Z} - K\right)_+\right], \quad Z \sim N(0,1)$$

$$= \cdots = \underline{\qquad} \cdot \underline{\qquad} \cdot$$