Generalized GBM (α, σ adapted processes)

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$$

Stock

Bank

Market

Discount factor $D_t = \exp\left(-\int_0^t R_s \, ds\right)$

Interest rate R_t (adapted process)

C_t = cash in Bank at time t

$$dC_t = +R_t C_t \, dt \quad \left(C_t = C_0 \exp\left(+\int_0^t R_s \, ds\right)\right)$$

Last time:

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P},}$ *where*

IOU explanation $\rightarrow$ $$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Theorem 10.5. *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s \, ds\Big) V_T\Big).$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Note $\frac{1}{D_t}\tilde{E}_t(D_T V_T) = \frac{1}{D_t}\tilde{E}_t\Big(\exp\Big(-\int_0^T R_s \, ds\Big) V_T\Big)$

$= \frac{1}{D_t}\tilde{E}_t\Big(\exp\Big(-\int_0^t R_s \, ds\Big) \exp\Big(-\int_t^T R_s \, ds\Big) V_T\Big) =$

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}.}$

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then* $\boxed{\tilde{\boldsymbol{E}}_s X_t = \frac{1}{Z_s} \boldsymbol{E}_s(Z_t X_t).}$

Proof. You will see this in the proof of the Girsanov theorem. □

Corollary 10.11. *M is martingale under $\tilde{\boldsymbol{P}}$ if and only if $\boxed{ZM}$ is a martingale under $\boldsymbol{P}$.*

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↳ Check: Say M is a $\tilde{P}$ mg.

Want to show ZM is a P mg.

Compute $$E_s(Z_t M_t) = Z_s \tilde{E}_s M_t \quad \text{(Lemma 10.10)}$$

$$= Z_s M_s \quad (\because M \text{ is a } \tilde{P} \text{ mg})$$

Conversely, say ZM is a P mg.

NTS M is a $\tilde{P}$ mg: $$\tilde{E}_s M_t = \frac{1}{Z} E_s(Z_t M_t) \overset{\text{mg}}{=} \frac{1}{\cancel{Z_s}} \cancel{Z_s} M_s$$

Theorem 10.12 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$ and let b be an adapted process.*

- *Define* $\tilde{W}_t = W_t + \int_0^t b_s\, ds$ *(i.e.* $d\tilde{W}_t = b_t\, dt + dW_t$*).*
- $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$*, where* $Z_t = \exp\left(-\int_0^t b_s\, dW_s - \frac{1}{2}\int_0^t |b_s|^2\, ds\right)$.

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

$d\tilde{W} = b\, dt + dW$

Proposition 10.13. $dZ_t = -Z_t b_t \cdot dW_t$.

Question 10.14. *Looks like Z is a martingale. Why did we assume it in Theorem 10.12?*

Let $X_t = \int_0^t b_s \, dW_s \longrightarrow d[X,X] = b_t^2 \, dt$

Let $f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t b_s^2 \, ds\right)$

Then $Z_t = f(t, X_t)$

$$\left|\begin{array}{l} \partial_t f = \exp(\ \ )\left(-\frac{1}{2} b_t^2\right) \\ \partial_x f = -\exp(\ \ ) \\ \partial_x^2 f = +\exp(\ \ ) \end{array}\right.$$

Ito

$\downarrow$

$$dZ = d\,f(t, X_t) = \partial_t f\,dt + \partial_x f\,dX + \frac{1}{2}\partial_x^2 f\,d[X,X]$$

$$= -\frac{1}{2}b_t^2 Z_t\,dt - Z_t\,dX_t + \frac{1}{2}Z_t\,d[X,X]$$

$$= \cancel{-\frac{1}{2}b_t^2 Z_t\,dt} - Z_t b_t\,dW + \cancel{\frac{1}{2}Z_t b_t^2\,dt}$$

$$= \underline{-Z_t b_t\,dW_t}$$

Note: $\int_0^t \sigma_s \, dW_s$ is a mg ONLY if $\boxed{E \int_0^t \sigma_s^2 \, ds < \infty}$

Even though $dZ_t = -\theta_t Z_t \, dW_t$

Z is only a mg if $E \int_0^t \theta_s^2 Z_s^2 \, ds < \infty$

(Not easy to check in general)

Idea behind the proof of Theorem 10.12.

To show $\tilde{W}$ is a BM under $\tilde{P}$

Use Levy's Criterion

$\tilde{W}$ is a BM under $\tilde{P}$ $\Leftrightarrow$ 0 $\tilde{W}$ is a cts process

1 $\tilde{W}$ is a mg

& 2 $[\tilde{W}, \tilde{W}]_t = t$

Check 1 & 2 so

$$d\tilde{W} = \underbrace{b\,dt}_{\text{does not affect QV}} + dW \quad \Bigg\} \Rightarrow d[\tilde{W},\tilde{W}] = dt \Rightarrow (2).$$

Check (1): Check $\tilde{W}$ is a mg under $\tilde{P}$

$\Longleftrightarrow$ $Z\tilde{W}$ is a mg under P.

By Product rule, $d(Z\tilde{W}) = Z\,d\tilde{W} + \tilde{W}\,dZ + d[Z,\tilde{W}]$

Recall $\boxed{dZ = -b Z_t\, dW}$ & $d\tilde{W} = b\,dt + dW$

$$\Rightarrow d(Z\tilde{W}) = Z_t(b_t\,dt + dW_t) - \tilde{W}_t b_t Z\,dW_t + (-b_t Z_t \cdot 1)\,dt$$

$$= Z(1-\tilde{W}b)\,dW_t$$

$\Rightarrow Z\tilde{W}$ is a mg. under P $\Rightarrow$ (1)

By Lenz $\Rightarrow$ $\boxed{\tilde{W} \text{ is a mg under } \tilde{P}}$

Theorem (Theorem 10.4). *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P},}$ *where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Proof of Theorem 10.4.

Want $D_t S_t$ to be a $\tilde{P}$ mg.

$D_t = \exp\left(-\int_0^t R_s \, ds\right) \Rightarrow dD_t = -R_t D_t \, dt$ (Finite 1st var)

$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW$

Compute $d(D_t S_t) = D_t \, dS_t + S_t \, dD_t + \underbrace{d[D_t, S_t]}_{0}$

$$= D_t\left(\alpha_t S_t\,dt + \sigma_t S_t\,dW\right) - R_t D_t S_t\,dt + 0$$

$$= D_t S_t(\alpha_t - R_t)\,dt + D_t \sigma_t S_t\,dW_t$$

$$= D_t \sigma_t S_t \underbrace{\left[\frac{\alpha_t - R_t}{\sigma_t}\,dt + dW\right]}_{d\tilde{W}}$$

Choose $\tilde{W} = W_t + \int_0^t \theta_s\,ds$, where $\theta_t = \dfrac{\alpha_t - R_t}{\sigma_t}$

Choose $\tilde{P}$ by Girsanov to make $\tilde{W}$ a BM.

$$\Rightarrow d\tilde{P} = Z_T \, dP \quad , \quad Z_T = \exp\left(- \int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right)$$

$$\Rightarrow d(D_t S_t) = D_t S_t \sigma_t \, d\tilde{W}$$

BM under $\tilde{P}$

$\Rightarrow D_t S_t$ is a mg under $\tilde{P}$. !!!

Theorem 10.15. *X_t represents the wealth of a self-financing portfolio if and only if $D_t X_t$ is a $\tilde{\boldsymbol{P}}$ martingale.*

Remark 10.16. The proof of the backward direction requires the *martingale representation theorem*, and is outlined on your homework.

Remark 10.17. This is the analog of Theorem 4.57

Proof of the forward direction.

Assume X is a self fin port

Show $D_t X_t$ is a $\tilde{P}$ mg.

① Self fin $\Rightarrow$ $dX_t = \Delta_t \, dS_t + R_t(X_t - \Delta_t S_t)\,dt$

② Compute dS in terms of $d\tilde{W}$

$$d\tilde{W}_t = \theta_t\,dt + dW \qquad \theta = \frac{\alpha - R}{\sigma}$$

$$dS = \alpha S\,dt + \sigma S\,\underline{dW}$$

$$= \alpha S\,dt + \sigma S\left(d\tilde{W} - \theta\,dt\right)$$

$$= \alpha S\,dt + \sigma S\,d\tilde{W} - (\alpha - R)\underline{S\,dt}$$

$$\boxed{dS = \underline{R} S\,dt + \sigma S\,d\tilde{W}}$$

Replaced $W \longrightarrow \tilde{W}$

& $\alpha \longrightarrow R$

Compute $d(D_t X_t) = D_t\, dX_t + X_t\, dD_t + d[D_t, X_t]$

$$= D_t\left(\Delta_t\, dS_t + R(X - \Delta_t S_t)\, dt\right) - R D_t X_t\, dt$$

$$= D_t \Delta_t \left(R_t S_t\, dt + \sigma_t S_t\, d\tilde{W}\right) - D_t \Delta_t R_t S_t\, dt$$

$$= D_t \Delta_t \sigma_t S_t\, d\hat{W} \Rightarrow D_t X_t \text{ is a } \tilde{P} \text{ mg!!}$$

Theorem (Theorem 10.5). *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t (D_T V_T) = \tilde{\boldsymbol{E}}_t \Big(\exp\Big(\int_t^T -R_s \, ds \Big) V_T) \Big) .$$

Remark 10.18. This is the analog of Proposition 4.1.

Proof of Theorem 10.5.

Replicate the sec.: Find a self fin port with payoff $= V_T$

Choose $X_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$.

1 NTS $X_T = V_T$ (true)

② NTS X_t = wealth of a self fin port.

$\iff D_t X_t$ is a $\tilde{P}$ mg.

Formula for X_t

Compute $\tilde{E}_s(D_t X_t) = \tilde{E}_s\left(\tilde{E}_t(D_T V_T)\right)$

$\underset{\text{tower}}{=} \tilde{E}_s(D_T V_T)$

$= D_s X_s$ (Formula for X_s).

11. Black Scholes Formula revisited

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

Theorem 11.1. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}, \qquad \tau = T - t. \tag{8.4}$$

Remark 11.2. This proves Proposition 8.8.

$D_t = e^{-rt}$

Proof: RNP Formula: $V_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$

$= e^{-r\tau} \tilde{E}_t\, g(S_T)$ $\leftarrow$ $(*)$

Under $\tilde{P}$: $dS_t = r S_t\, dt + \sigma S_t\, d\tilde{W}_t$

$\Rightarrow S$ is $GBM(r, \sigma)$ under $\tilde{P}$.

$$\Rightarrow S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right)$$

$$\Rightarrow S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T\right)$$

$$\Rightarrow S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

Substitute in $(*)$

$$V_t = e^{-r\tau}\,\tilde{E}_t\, g(S_T)$$

$$= e^{-r\tau}\,\tilde{E}_t\, g\left[\underbrace{S_t}_{\mathcal{F}_t \text{ meas}} \exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,\overbrace{\frac{(\tilde{W}_T-\tilde{W}_t)}{\sqrt{\tau}}}^{N(0,T-t)\ \&\ \mathcal{F}_t \text{ indep}}\right)\right]$$

Indep lemma

$$e^{-r\tau}\int_{-\infty}^{\infty} g\left(S_t \exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y\right)\right) \frac{e^{-y^2/2}\, dy}{\sqrt{2\pi}}$$

$**$

QED!!

Theorem 11.3 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy, \tag{8.7}$$

is the CDF of a standard normal variable.

Remark 11.4. This proves Corollary 8.9.

Substitute $g(x) = (x-K)^+$ in $(**)$

$$c(t,x) = e^{-r\tau}\int_{-\infty}^{\infty}\Big(x\,e^{(r-\sigma^2/2)\tau + \sigma\sqrt{\tau}y} - K\Big)^+ e^{-y^2/2}\,\frac{dy}{\sqrt{2\pi}} \quad (***)$$

$$\text{Salve} \quad \times e^{\left(r-\sigma^2/2\right)\tau + \sigma\sqrt{\tau}\, y} = K$$

$$\Rightarrow \left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y = \ln\left(\frac{K}{x}\right) = -\ln\left(\frac{x}{K}\right)$$

$$\Rightarrow y = -\frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\tau\right)$$

$$= -d_{-}$$

$$\Rightarrow (***)$$

$$C(t,x) = e^{-r\tau} \int_{-d_-}^{\infty} \left(x e^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y} - K \right) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

& simplify ...