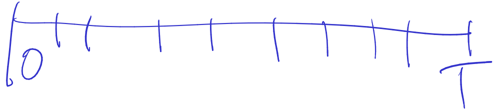
9. Multi-dimensional Itô calculus



- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \, ,$$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t \, dY_t$.

$$4ab = (a+b)^2 - (a-b)^2$$

Lemma 9.3. $\underbrace{[X, Y]_T}_{\text{Joint QV}} = \frac{1}{4}(\underbrace{[X + Y, X + Y]_T}_{\text{QV of } X+Y} - [X - Y, X - Y]_T)$

QV of $X - Y$

Proposition 9.4. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X, Y]_t = d[M, N]_t$.*

Remark 9.5. Recall, all processes are implicitly assumed to be *adapted* and *continuous*.

Corollary 9.6. *If X is a semi-martingale and B has bounded variation then $[X, B] = 0$.*

① $[X,B] = \frac{1}{4}\Big(\underbrace{[X+B, X+B]}_{[X,X]} - \underbrace{[X-B, X-B]}_{[X,X]}\Big) = 0$

② $X = X_0 + C + M$

& $B = B_0 + (B - B_0) + 0$

$[X,B] = [M, 0] = 0$

Remark 9.7 (Chain rule). If X, Y are differentiable functions of t, then

$$d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t$$

Remark 9.8 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_x f = \frac{\partial f}{\partial x}$, $\partial_y f = \frac{\partial f}{\partial y}$.

$$f = f(t, x, y)$$

$$\frac{d}{dt} f(t, X_t, Y_t) \overset{\text{Chain Rule}}{=} \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(\ )\frac{dX_t}{\cancel{dt}} + \partial_y f(\ )\frac{dY_t}{\cancel{dt}}$$

If X & Y are not diff, then there is an extra term:

$$\frac{1}{2}\left[\partial_x^2 f \, d[X,X] + \partial_y^2 f \, d[Y,Y] + 2\partial_x \partial_y f \, d[X,Y]\right]$$

Theorem 9.9 (Two-dimensional Itô formula).

- *Let X, Y be two processes.*
- *Let $f = f(t, x, y)$ be a $C^{1,2}$ function. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in both x, and y.*
  - ▷ *All the above partial derivatives are continuous. Then:*

from chain rule

$$d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t$$

$2\partial_x\partial_y f$

$$+ \frac{1}{2}\left[\partial_x^2 f(t, X_t, Y_t)\, d[X, X]_t + \partial_y^2 f(t, X_t, Y_t)\, d[Y, Y]_t + 2\partial_x\partial_y f(t, X_t, Y_t)\, d[X, Y]_t\right]$$

Remark 9.10. As with the 1D Itô, will drop the arguments (t, X_t, Y_t). *Remember they are there.*

Remark 9.11 (Integral form of Itô's formula).

$$f(T, X_T, Y_T) - f(0, X_0, Y_0) = \int_0^T \partial_t f\, dt + \int_0^T \partial_x f\, dX_t + \int_0^T \partial_y f\, dY_t$$

$$+ \frac{1}{2}\int_0^T \left(\partial_x^2 f\, d[X, X]_t + \partial_y^2 f\, d[Y, Y]_t + 2\partial_x\partial_y f\, d[X, Y]_t\right)$$

Intuition behind Theorem 9.9.

(1) 1D Ito: $\partial_x^2 f \, d[X,X]$ came from taylor

$$f(x+h) = f(x) + h f'(x) + \frac{1}{2} h^2 f''(x) + \text{small}$$

(2) In 2D: Taylors formula is

$$f(x+h, y+k) \approx f(x,y) + h\,\partial_x f(x,y) + k\,\partial_y f(x,y)$$

$$+ \frac{1}{2}\Big[\underbrace{\partial_x^2 f\, h^2}_{QV(X)} + \underbrace{\partial_y^2 f\, k^2}_{QV(Y)} + \underbrace{2\partial_x\partial_y f\, hk}_{\text{Joint QV.}}\Big]$$

Proposition 9.12 (Product rule). $d(XY)_t = X_t\,dY_t + Y_t dX_t + \boxed{d[X,Y]_t}$

If f & g are diff fnc,

$$\frac{d}{\cancel{dt}}(fg) = \frac{df}{\cancel{dt}}\,g + f\,\frac{dg}{\cancel{dt}}$$

Check product rule using Ito!

Let $f(x,y) = xy$

$\partial_t f = 0$

$\partial_x f = y \qquad \partial_y f = x$

$\partial_x^2 f = 0 \qquad \partial_y^2 f = 0$

$\partial_x \partial_y f = 1$

By Itô: $d(X_t Y_t) = d(f(X_t, Y_t))$

$$= \partial_t f \, dt + \partial_x f \, dX_t + \partial_y f \, dY_t + \frac{1}{2}\left[\partial_x^2 f \, d[X,X] + \partial_y^2 f \, d[Y,Y] + 2\partial_x\partial_y f \, d[X,Y]\right]$$

$$= 0 + Y_t \, dX_t + X_t \, dY_t + \frac{1}{\cancel{2}} \cancel{2} \, 1 \, d[X,Y].$$

$\Rightarrow$ Product Rule.

To use the multi-dimensional Itô formula, we need to compute joint quadratic variations.

Proposition 9.13. *Let M, N be continuous martingales, with $\boldsymbol{E} M_t^2 < \infty$ and $\boldsymbol{E} N_t^2 < \infty$.*

(1) $MN - [M,N]$ is also a continuous martingale.

(2) Conversely if $MN - B$ is a continuous martingale for some continuous adapted, bounded variation *process B with $B_0 = 0$, then $B = [M,N]$.*

Proof.

NTS $MN - [M,N]$ is a mg

$$d(MN - [M,N]) \overset{\text{Product rule}}{=} M\,dN + N\,dM + d[M,N] - d[M,N]$$

$$= \underbrace{M\,dN + N\,dM}_{Mg}.$$

Proposition 9.14. *(1)* (Symmetry) $[X, Y] = [Y, X]$

(2) (Bi-linearity) *If* $\alpha \in \mathbb{R}$, X, Y, Z *are semi-martingales,* $[X, Y + \alpha Z] = [X, Y] + \alpha[X, Z]$.

Proof.

Proposition 9.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \int_0^t \sigma_s \, dM_s$ *and* $Y_t = \int_0^t \tau_s \, dN_s$.
- *Then* $\boxed{[X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N]_s.}$

$\leftarrow$ Gen of our rule to compute QV of Ito int.

Remark 9.16. In differential form, if $dX_t = \sigma_t \, dM_t$ and $dY_t = \tau_t \, dN_t$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

$$\Delta_i X = X_{t_{i+1}} - X_{t_i} \approx \sigma_{t_i} \Delta_i M$$

$$\Delta_i Y = \qquad \approx \tau_{t_i} \Delta_i N$$

$$\implies \underbrace{\Delta_i X \, \Delta_i Y}_{d[X,Y]} \approx \sigma_{t_i} \tau_{t_i} \underbrace{\Delta_i M \, \Delta_i N}_{\approx d[M,N]}$$

Proposition 9.17. *If M, N are continuous martingales, $\boldsymbol{E}M_t^2 < \infty$, $\boldsymbol{E}N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 9.18 (Warning). Independence implies $\boldsymbol{E}(M_t N_t) = \boldsymbol{E}M_t \boldsymbol{E}N_t$. But it *does not* imply $\boldsymbol{E}_s(M_t N_t) = \boldsymbol{E}_s M_t \boldsymbol{E}_s N_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof. → Tricky

Intuition: $[M,N]_T = \lim_{\|P\| \to 0} \sum \Delta_i M \, \Delta_i N$

Trick: Compute $E\left(\sum \Delta_i M \, \Delta_i N\right)^2 \xrightarrow[\|P\| \to 0]{\text{Will show}} 0$

(1) $$E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2 = E\sum (\Delta_i M)^2(\Delta_i N)^2 + 2E\sum_{i<j} (\Delta_i M)(\Delta_j M)(\Delta_i N)(\Delta_j N)$$

$$= \underbrace{\sum E(\Delta_i M)^2 E(\Delta_i N)^2} + 2\sum_{i<j} \underbrace{E(\Delta_i M\,\Delta_j M)\,E(\Delta_i N\,\Delta_j N)}_{\substack{\text{Use tower \& } M \text{ is a mg} \\ \text{check } = 0}}$$

$$\le \underbrace{\left[\max_i E(\Delta_i M)^2\right]}_{\substack{\Big\downarrow \, \|P\| \to 0 \\ 0}} \cdot \underbrace{\sum_i E(\Delta_i N)^2}_{\substack{\Big\downarrow \, \|P\| \to 0 \\ E[N,N]}}$$

($\because$ M is cts!)

Remark 9.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- Let $M_t = \int_0^t \mathbf{1}_{\{W_s < 0\}} \, dW_s$
- Let $N_t = \int_0^t \mathbf{1}_{\{W_s \geqslant 0\}} \, dW_s$

$$M + N = \int_0^t \left(\mathbf{1}_{\{W_s < 0\}} + \mathbf{1}_{\{W_s \geq 0\}} \right) dW = \int_0^t 1 \, dW_s = W_t$$

$$d[M, N] = \mathbf{1}_{\{W_s < 0\}} \mathbf{1}_{\{W_s \geq 0\}} \, d[W, W]_t = 0 \, dt$$

$$\Rightarrow \boxed{M_t + N_t = W_t}$$

Question 9.20. *Let W^1 and W^2 be two independent Brownian motions, and let $W = (W^1, W^2)$. Define the process X by $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. $\boxed{\text{Is } X \text{ a martingale?}}$*

$W_t^1 \longrightarrow 1^{st}$ BM at time t

$W_t^2 \longrightarrow 2^{nd}$ BM at time t

$W = (W^1, W^2)$ $\qquad$ $|W| = \sqrt{(W^1)^2 + (W^2)^2}$

$X_t = \ln\left[(W_t^1)^2 + (W_t^2)^2\right]$ ✓

Is X a mg?

Itô: $f(t,x,y) = \ln(x^2+y^2)$

Compute $\partial_x f$, $\partial_y f$, $\partial_x^2 \dots$

Compute $dX_t = d f(t, W_t^1, W_t^2)$

$$\overset{\text{Itô}}{=} \partial_x f \, dW_t^1 + \partial_y f \, dW_t^2 + \frac{1}{2}\Big[\partial_x^2 f \, \underbrace{d[W^1,W^1]}_{dt} + \partial_y^2 f \, \underbrace{d[W^2,W^2]_t}_{dt} + 2\partial_x\partial_y f \, d[W^1,W^2]\Big]$$

∂ (indep)

$$= \partial_x f \, dW^1_t + \partial_y f \, dW^2_t + \frac{1}{2}\left[\partial_x^2 f + \partial_y^2 f\right] dt$$

Can compute & check $\partial_x^2 f + \partial_y^2 f = 0$ (when $f(x,y) = \ln(x^2+y^2)$)

$$\boxed{\Rightarrow dX_t = \partial_x f \, dW^1_t + \partial_y f \, dW^2_t} \Rightarrow X \text{ is a mg??}$$

Claim: Even though there are no dt terms here X is NOT a mg!!

(Reason: ________________________)

10. Risk Neutral Pricing

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t\, dt + \sigma_t S_t\, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $\boxed{D_t = \exp\Big(-\int_0^t R_s\, ds\Big)}$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes $1 in the bank at time t.

Recall: C_t = cash in bank at time t

$$\partial_t C_t = R_t C_t \Rightarrow C_t = C_0 \underbrace{\exp\Big(\int_0^t R_s\, ds\Big)}$$

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P},}$ *where*

Market Price of risk

$$Z_T = \exp\Big(-\int_0^T \theta_t\, dW_t - \frac{1}{2}\int_0^T \theta_t^2\, dt \Big)\,, \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}\,.}$$

Theorem 10.5. *Any security can be replicated. If a security pays* V_T *at time* T, *then the arbitrage free price at time* t *is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\, ds\Big)\, V_T\Big)\,.$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$ and prove both the above theorems later.

Disc time: RNP formula $V_n = \dfrac{1}{D_n}\tilde{E}_n(D_N V_N)$

(disc time version of

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}.}$

(Need $Z_T > 0$ to ensure $P(A) = 0 \iff \tilde{P}(A) = 0$)

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then $\tilde{\boldsymbol{E}}_s X_t = \frac{1}{Z_s} \boldsymbol{E}_s(Z_t X_t)$.*

RN cond Exp.

Proof. You will see this in the proof of the Girsanov theorem. ∎

Corollary 10.11. *M is martingale under $\tilde{\boldsymbol{P}}$ if and only if ZM is a martingale under $\boldsymbol{P}$.*

Q: $\tilde{P}(A) \in [0,1]$

Check: ① $\tilde{P}(A) = E(\mathbb{1}_A Z_T) \geq 0$ $(\because Z_T > 0)$

② $\tilde{P}(A) = E(\mathbb{1}_A Z_T) \leq E Z_T = 1$

Q: Need $\tilde{P}(\Omega) = 1$

1 Note $\tilde{P}(\Omega) = E(\mathbb{1}_\Omega Z_T) = E Z_T = 1$

Prop: $X_t \longrightarrow \mathcal{F}_t$ meas

$$Z_t = E_t Z_T$$

Then $\tilde{E}_s X_t = \frac{1}{Z_s} E_s(Z_t X_t)$

Claim: M is a $\tilde{P}$ mg $\iff$ ZM is a P mg.

① Say ZM is a P-mg

Compute $\tilde{E}_s M_t = \frac{1}{Z_s} E_s(Z_t M_t)$

$= \frac{1}{Z_s}(Z_s M_s)$ $(\because ZM$ is a P-mg$)$

$= M_s \Rightarrow M$ is a $\tilde{P}$ mg.