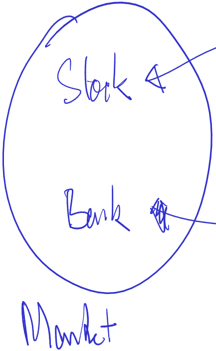
Last time: Cts time market



GBM(α, σ) : $dS_t = \alpha S_t dt + \sigma S_t dW_t$

Int rate r : $dC_t = r C_t$ $\left(\iff C_t = C_0 e^{rt}\right)$

B.S. PDE:

Security Pays $V_T = g(S_T)$ at maturity T

Thm 1 If sec can be rep & $X_t = AFP = f(t, S_t)$

then f satisfies

PDE

1. $\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f$

2. $f(x, T) = g(x)$ (Terminal cond)

3. $f(0, t) = g(0) e^{-r(T-t)}$ (Boundary cond)

Thm 2: If f solves (1) (2) & (3)

then the sec can be rep

& X_t = wealth of R. Port = AFP $= f(t, S_t)$

Last time: Proved (1) $\to$

X self fin if $dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt$

Know $X_t = f(t, X_t)$ is self fin

Compute dX_t by self fin
& $d\,f(t, S_t)$ by Itô } equate & get B.S. PDE.

Thm 2: Assume f solves the BS PDE (1), (2), (3)

Claim: $f(t, S_t)$ = wealth of the R. Port

Last time: Let X_t = wealth of a self fin Port with
$$X_0 = f(0, S_0)$$

~~*Proof of Theorem 8.4 (without discounting).*~~

$$\& \quad \Delta_t = \partial_x f(t, S_t)$$

$$dX_t = \Delta_t \, dS + r(X_t - \Delta_t S_t)\, dt$$

$$=$$

Let $Y_t = f(t, S_t)$ (Want to show $X_t = Y_t$)

If we know $X_t = Y_t$ then $X_T = Y_T = f(T, S_T) = g(S_T) = V_T$

$\Rightarrow X_t = f(t, S_t) =$ wealth of R port $= AFP$

$\rightarrow$ Compute dY_t by Itô: $dY_t = ? \, dt + \ldots$

Then simplify & write $d(Y - X)$

$$d(Y_t - X_t) = \Big(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2} \partial_x^2 f$$

$$- (\alpha - r)\Delta_t S_t - r X_t\Big) dt + 0\, dW$$

$$\boxed{\text{Recall } \partial_x f(t, S_t) = \Delta_t}$$

$$= \Big(\underbrace{\partial_t f + r S_t \partial_x f + \frac{\sigma^2 S^2}{2} \partial_x^2 f}_{rf} - r X_t\Big) dt$$

$$\Rightarrow d(Y_t - X_t) = \big(r f(t, S_t) - r X_t\big) dt = r(Y_t - X_t)\, dt$$

$$\Rightarrow \quad Y_t - X_t = (Y_0 - X_0)e^{rt}$$

$$= 0 \quad \Rightarrow \text{done!}$$

$$\boxed{\begin{array}{l}\text{Remember by choice} \\ X_0 = f(0, S_0) = Y_0\end{array}}$$

Remark 8.12. The arbitrage free price does not depend on the mean return rate!

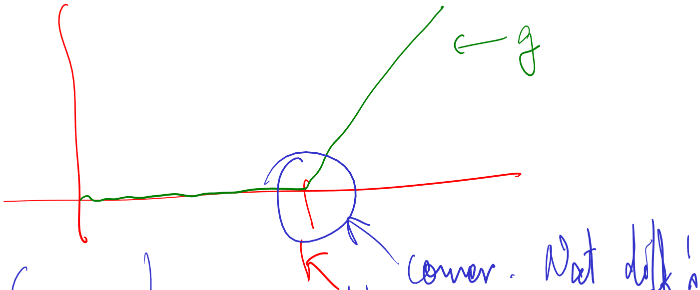
$$dS_t = \alpha S\,dt + \sigma S\,dW_t$$

mean ret rate

$$\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f \quad \leftarrow \text{no } \alpha!$$

Question 8.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (8.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

$g(x) = (x - K)^+$



Our Pf of BS. is valid (even if g is not diff)

because: We only need Ito on $f(t, S_t)$ with $t < T$

and for $t < T$, f is $C^{1,2}$ (1 der in t & 2 der in x)

Proposition 8.14 (Put call parity). *Consider a European put and European call with the same strike K and maturity T.*

- ▷ $c(t, S_t) =$ *AFP of call (given by (8.5))*
- ▷ $p(t, S_t) =$ *AFP of put.*

Then $\boxed{c(t,x) - p(t,x) = x - Ke^{-r(T-t)}}$, *and hence* $\boxed{p(t,x) = Ke^{-r(T-t)} - x - c(t,x)}$.

Consider a put

$\left.\begin{array}{l} +1 \text{ call} \\ -1 \text{ put} \end{array}\right\} \rightarrow$ Payoff : $(S_T - K)^+ - (K - S_T)^+$

$= \underbrace{S_T - K}$ ← Forward Contract.

AFP of F.C. $= S_t - Ke^{-r(T-t)}$

$\Rightarrow c(t, S_t) - p(t, S_t) = S_t - Ke^{-r(T-t)}$

8.3. **The Greeks.** Let $c(t,x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

B.S. Formula.

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad \tau = T - t\,.$$

Definition 8.15. The $\boxed{\textit{delta}}$ is $\partial_x c$.

Remark 8.16 (Delta hedging rule). $\boxed{\Delta_t = \partial_x c(t, S_t).}$

Proposition 8.17. $\partial_x c = N(d_+)$

$$N(y) = \text{CDF of Normal} = \int_{-\infty}^{y} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$$

$$c = xN(d_+) - Ke^{-r\tau}N(d_-)$$

$$\partial_x c = N(d_+) + xN'(d_+)\frac{1}{\sigma\sqrt{\tau}\,x} - Ke^{-r\tau}N'(d_-)\frac{1}{\sigma\sqrt{\tau}\,x}$$

$$= N(d_+) + x\left(\frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\right)\frac{1}{\sigma\sqrt{\tau}x} - Ke^{-r\tau}\left(\frac{e^{-d_-^2/2}}{\sqrt{2\pi}}\right)\frac{1}{\sigma\sqrt{\tau}x}$$

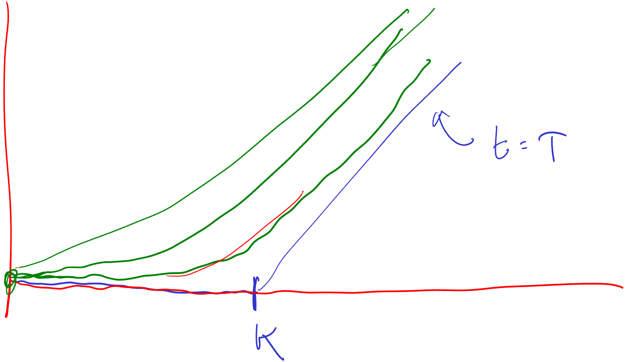
turns out to be 0

set $= N(d_+)$.

Definition 8.18. The *Gamma* is $\partial_x^2 c$ and is given by $\partial_x^2 c = \dfrac{1}{x\sigma\sqrt{2\pi\tau}}\exp\left(\dfrac{-d_+^2}{2}\right)$.

Definition 8.19. The *Theta* is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \dfrac{\sigma x}{2\sqrt{\tau}}N'(d_+)$

$$\partial_x^2 c = \partial_x\left(\partial_x c\right) = \partial_x\left(N(d_+)\right) = N'(d_+)\,\frac{1}{\sigma\sqrt{\tau}\,x}$$



$c(t,x)$ as $t \to T$

Proposition 8.20. *(1) c is increasing as a function of x.*

(2) c is convex as a function of x.

(3) c is decreasing as a function of t.

(1) $\rightarrow$ Since $\partial_x C = N(d_+) > 0$

(2) $\partial_x^2 C = \text{Gamma} > 0$

(3) $\partial_t C = \text{Theta} < 0$

Remark 8.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Pout Payoff $= (x-K)^+$

R port: Δ_t shares of stock
Rest cash.

Know $\Delta_t = \partial_x c(t, S_t)$

$c(t, S_t) - \Delta_t S_t = c(t, S_t) - \partial_x c(t, S_t) S_t$

Put $x = S_t$

Cash balance $= c(t,x) - x\,\partial_x c(t,x)$

$$= \cancel{x N(d_+)} - K e^{-r\tau} N(d_-) - \cancel{x N(d_+)}$$

$$= -\underline{K e^{-r\tau} N(d_-)} < 0.$$

Q: What is Δ_T (for the R part of Eur call)

$$\Delta_T = \begin{cases} 1 & S_T \geq K \\ 0 & S_T \leq K. \end{cases}$$

$x = S_t$.

$$\Delta_t = \partial_x c(t, x) = N(d_+)$$

$\tau = T - t$

$$d_+ = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right)$$

As $t \to T$, $\tau \to 0$

$$\lim_{t \to T} d_+ = \begin{cases} +\infty & x > K \\ -\infty & x < K \end{cases}$$

$$\Rightarrow \lim_{t \to T} \partial_x c = \begin{cases} N(+\infty) = 1 & x > K \\ N(-\infty) = 0 & x < K. \end{cases}$$

Remark 8.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x) -$ tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

Port value when spot price is $x = c(t,x) + x_0 \partial_x c(t,x_0) - c(t,x_0)$

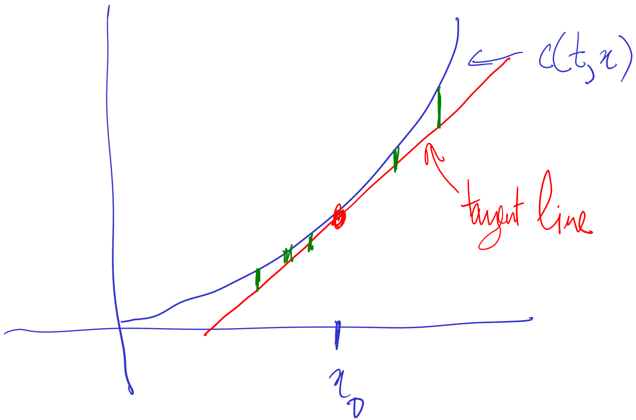
$$- x\, \partial_x c(t,x_0)$$

$$= c(t,x) - \Big[\underbrace{(x - x_0)\partial_x c(t,x_0) + c(t,x_0)}_{\text{tangent line}} \Big]$$

u

Portfolio value :

$c(t,x) \longrightarrow$ tgt line



9. Multi-dimensional Itô calculus

- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$\boxed{[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),}$$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t\, dY_t$.

QV: $[X, X] = \lim_{\|P\| \to 0} \underbrace{\sum (X_{t_{i+1}} - X_{t_i})^2}$

Y

$(X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$

Lemma 9.3. $\boxed{[X,Y]_T} = \frac{1}{4}([X+Y,X+Y]_T - [X-Y,X-Y]_T)$

QV of $X+Y$

QV of $X-Y$.

$$(a+b)^2 - (a-b)^2 = \underline{4ab}.$$

$$\Delta_i X = X_{t_{i+1}} - X_{t_i}$$

$$\Delta_i Y = Y_{t_{i+1}} - Y_{t_i}$$

$$\Delta_i X \, \Delta_i Y = \frac{1}{4}\left(\left[\Delta_i (X+Y)\right]^2 - \left[\Delta_i (X-Y)\right]^2 \right)$$