Midterm Problem

2. $X = \begin{cases} 1 & \text{Head} \quad 1/2 \\ 0 & \text{Tail} \quad 1/2 \end{cases}$ $\quad Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$

$\mathbb{E}[e^{\lambda XY} \mid X] = g(X)$ where $g(x) = \mathbb{E}[e^{\lambda x Y}]$ (Ind lem)

$= e^{\frac{\lambda^2 X^2}{2}}$ $\quad = e^{\frac{\lambda^2 x^2}{2}}$

- Common Mistake : $\mathbb{E}[e^{\lambda \cdot 1 Y}] \cdot \mathbb{P}(X=1) + \mathbb{E}[e^{\lambda \cdot 0 \cdot Y}] \, \mathbb{P}(X=0) = \mathbb{E}[e^{\lambda Y}]$

- Joint pdf : $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$. ✓

3. $X_t = W_t^2 + \underbrace{\int_0^t \sqrt{1+s^2}\, dW_s + \int_0^t \sqrt{3+W_s^4}\, ds}_{=: Y_t}$, $\quad \langle X, X \rangle_t = ?$

$dX_t = \underline{dW_t^2} + \sqrt{1+t^2}\, dW_t + \sqrt{3+W_t^4}\, dt$

$= 2W_t dW_t + dt +$ 〃

$= (2W_t + \sqrt{1+t^2})\, dW_t + (1 + \sqrt{3+W_t^4})\, dt$

$\Rightarrow$ $\langle X, X \rangle_t$

$= \int_0^t (2W_s + \sqrt{1+s^2})^2\, ds$

• Common Mistake : $\langle X, X\rangle_t = \langle W^2, W^2\rangle_t + \langle Y, Y\rangle_t$

$$= \int_0^t 1 + \left(\sqrt{1+s^2}\right)^2 ds \qquad \times$$

• $\langle X, X\rangle_t = \mathbb{E}\int_0^t \left(2W_s + \sqrt{1+s^2}\right)^2 ds$, $\mathbb{E}[X_t^2] = ?$

5. $X_t = \int_0^t e^{-W_r} dW_r$, $\mathbb{E}_s[X_t^2] = ?$

• <u>Common Mistake :</u> $\mathbb{E}_s[X_t^2] = \mathbb{E}_s\left[\left(\int_0^t e^{-W_r} dW_r\right)^2\right]$

$$\neq \mathbb{E}_s\left[\int_0^t e^{-2W_r} dr\right]$$

$$\mathbb{E}_s[X_t^2] = \mathbb{E}_s\left[\int_0^t e^{-2W_r} dr\right] + \text{" } \underbrace{(\quad\quad)}\text{ "}$$

$W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

• $\mathbb{E}_s\left[\int_s^t e^{-W_r} dr\right] \neq \mathbb{E}\left[\int_s^t e^{-W_r} dr\right]$

$$\mathbb{E}_s\left[\int_s^t e^{-(W_r - W_s)} dr\right] = \int_s^t \mathbb{E}_s\left[e^{-(W_r - W_s)}\right] dr = \int_s^t \mathbb{E}\left[e^{-(W_r - W_s)}\right] dr$$

BS Model $\quad dS_t = \alpha S_t dt + \sigma S_t dW_t , \quad C_t = C_0 e^{rt}$

Def $\quad X$ is a self-financing portfolio if

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt \quad \text{for some adapted } \Delta_t$$

Note $\quad X_0 , \Delta_t$ are given to us.

Is it possible to construct a self-fin port X that corresponds to (X_0, Δ_t) ?

Ans $\quad$ Yes...

Def $\quad V_T$: A payoff of an option with maturity T.

X is a replicating portfolio if

① X is self-fin

② $X_T = V_T$

Thm 8.3. $V_T = g(S_T)$.

If this option can be replicated by a replicating portfolio $f(t, S_t)$,

then f satisfies

$$\begin{cases} ① \ \partial_t f + rx\,\partial_x f + \dfrac{\sigma^2 x^2}{2}\partial_x^2 f = rf, & \forall x > 0, \ \forall 0 < t < T \\ ② \ f(t,0) = g(0)\,e^{-r(T-t)}, & \forall 0 \le t \le T \\ ③ \ f(T,x) = g(x), & \forall x \ge 0 \end{cases}$$

→ BS PDE

prop 8.8 $f(t,x) = e^{-r\tau} \displaystyle\int_{-\infty}^{\infty} g\left(x\, e^{(r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy, \quad \tau = T - t$

why ?

③ : $f(T, x) = \int_{-\infty}^{\infty} g(x) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = g(x) \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$

$= g(x)$

② : $f(t, 0) = e^{-r\tau} \int_{-\infty}^{\infty} g(0) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = e^{-r\tau} \cdot g(0)$

$= e^{-r(T-t)} g(0).$

① : $g(x) = (x-K)_+$

NOT EASY ....

Note prop 8.8 can be proved using "Girsanov Thm" + "RN"

Ex European call option, $g(x) = (x-K)_+$, AFP $= V_t$?

By Thm 8.3, prop 8.8 $V_t = f(t, S_t)$ where

$$f(t,x) = e^{-r\tau}\int_{-\infty}^{\infty}\left(x e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}-k\right)_+ \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}, \quad \tau = T-t.$$

- $x e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}-k \geq 0 \iff e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} \geq \frac{k}{x}$

$$\iff (r-\tfrac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y \geq \log\left(\tfrac{k}{x}\right)$$

$$\iff \sigma\sqrt{\tau}y \geq \log\left(\tfrac{k}{x}\right)-(r-\tfrac{\sigma^2}{2})\tau$$

$$\iff y \geq \frac{\log\left(\frac{k}{x}\right)-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$$\iff y \geq -\left[\frac{\log\left(\frac{x}{k}\right)+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right]$$

$$\iff y \geq -d_-$$

$$f(t,x) = e^{-r\tau}\int_{-d_-}^{\infty} x e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}-k \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}$$

$$= e^{-r\tau}\int_{-d_-}^{\infty} x\, e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}\cdot\frac{e^{-y^2/2}dy}{\sqrt{2\pi}} - ke^{-r\tau}\int_{-d_-}^{\infty}\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}$$

$$= e^{-r\tau}\cdot x\cdot e^{(r-\frac{\sigma^2}{2})\tau}\cdot\int_{-d_-}^{\infty} e^{\sigma\sqrt{\tau}y}\frac{e^{-y^2/2}dy}{\sqrt{2\pi}} - ke^{-r\tau}\,\mathbb{P}(Z\geq -d_-)$$

where $Z \sim N(0,1)$.

$$= \qquad '' \qquad -ke^{-r\tau}\,\mathbb{P}(Z\leq d_-)$$

$$= \qquad '' \qquad -ke^{-r\tau}N(d_-)$$

$$= e^{-r\tau}\cdot x\cdot e^{(r-\frac{\sigma^2}{2})\tau}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2+\sigma\sqrt{\tau}y}\,dy - ke^{-r\tau}N(d_-)$$

$$= e^{-r\tau}\cdot x\cdot e^{(r-\frac{\sigma^2}{2})\tau}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\cdot e^{\frac{1}{2}\sigma^2\tau}\,dy - ke^{-r\tau}N(d_-)$$

$$= x\cdot\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}dy - ke^{-r\tau}N(d_-)$$

$$= x \cdot \int_{-d_- - \sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - Ke^{-r\tau} N(d_-)$$

$$= x \cdot \mathbb{P}(Z \geq -d_- - \sigma\sqrt{\tau}) - Ke^{-r\tau} N(d_-)$$

$$= x \cdot N(d_- + \sigma\sqrt{\tau}) - Ke^{-r\tau} N(d_-)$$

$$= x \cdot N(d_+) - Ke^{-r\tau} N(d_-). \qquad \blacksquare$$

$$\left(d_+ = \frac{\log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \qquad d_- = \frac{\log(\frac{x}{K}) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \right)$$

<u>EX</u> Digital (Binary) option. $g(x) = \mathbb{1}_{(x \geq K)} = \begin{cases} 1 & x \geq K \\ 0 & x < K \end{cases}$

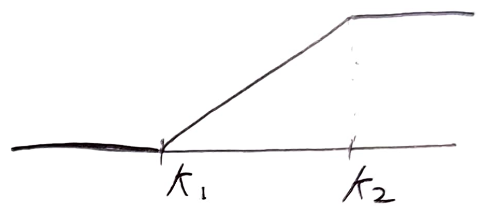
$$f(t,x) = e^{-r\tau} \int_{-\infty}^{\infty} \mathbb{1}\left(x e^{(r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y} \geq K \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

☆ $x e^{(r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y} \geq K \iff y \geq -d_-$ (from previous EX)

$$= e^{-r\tau}\int_{-d_-}^{\infty} 1 \cdot \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}} + e^{-r\tau}\int_{-\infty}^{-d_-} 0 \cdot \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}$$

$$= e^{-r\tau}\, P(Z \geq -d_-) = e^{-r\tau} N(d_-)$$

<u>Ex</u> (Bull-Spread) $g(x) =$



$$= \begin{cases} 0 & x < K_1 \\ x - K_1 & K_1 \leq x < K_2 \\ K_2 - K_1 & x \geq K_2 \end{cases}$$

$$f(t,x) = e^{-r\tau}\int_{-\infty}^{\infty} g\left(x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y}\right) \cdot \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}$$

- $x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y} \leq K_1 \iff y \leq -d_-^{(1)}$
- $K_1 \leq x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y} \leq K_2 \iff -d_-^{(1)} \leq y \leq -d_-^{(2)}$
- $K_2 \leq x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y} \iff y \geq -d_-^{(2)}$

$$f(t,x) = \underline{e^{-r\tau}\int_{-\infty}^{-d_-^{(1)}} 0 \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy} + \underbrace{e^{-r\tau}\int_{-d_-^{(1)}}^{-d_-^{(2)}} \left(x e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} - K_1\right)\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}}_{=: \mathrm{I}}$$

$$+ \underbrace{e^{-r\tau}\int_{-d_-^{(2)}}^{\infty} (K_2-K_1)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy}_{=: \mathrm{II}}$$

$$\mathrm{II} = e^{-r\tau}\cdot(K_2-K_1)\cdot\int_{-d_-^{(2)}}^{\infty}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = e^{-r\tau}(K_2-K_1)N(d_-^{(2)})$$

$$\mathrm{I} = \underbrace{e^{-r\tau}\int_{-d_-^{(1)}}^{-d_-^{(2)}} x e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}}_{=: \mathrm{III}} - \underline{K_1 e^{-r\tau}\int_{-d_-^{(1)}}^{-d_-^{(2)}}\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}}$$

$$= \quad '' \quad - K_1 e^{-r\tau}\mathbb{P}(-d_-^{(1)} \le Z \le -d_-^{(2)})$$

$$= \quad '' \quad - K_1 e^{-r\tau}\mathbb{P}(d_-^{(2)} \le Z \le d_-^{(1)})$$

$$= \quad '' \quad - K_1 e^{-r\tau}(N(d_-^{(1)}) - N(d_-^{(2)}))$$

$$=$$

$$\text{III} = e^{-r\tau} \int_{-d_-^{(1)}}^{-d_-^{(2)}} x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

$$= e^{-r\tau} \cdot x \cdot e^{(r-\frac{\sigma^2}{2})\tau} \int_{-d_-^{(1)}}^{-d_-^{(2)}} \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{\tau}y - y^2/2} dy$$

$$= e^{-r\tau} \cdot x \cdot e^{(r-\frac{\sigma^2}{2})\tau} \cdot e^{\frac{1}{2}\sigma^2\tau} \int_{-d_-^{(1)}-\sigma\sqrt{\tau}}^{-d_-^{(2)}-\sigma\sqrt{\tau}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$= x \cdot \left(N(d_-^{(1)} + \sigma\sqrt{\tau}) - N(d_-^{(2)} + \sigma\sqrt{\tau}) \right)$$

$$= x \cdot \left(N(d_+^{(1)}) - N(d_+^{(2)}) \right)$$

$$\therefore f(t,x) = x\left(N(d_+^{(1)}) - N(d_+^{(2)})\right) - K_1 e^{-r\tau}\left(N(d_-^{(1)}) - N(d_-^{(2)})\right)$$

$$+ e^{-r\tau}(K_2 - K_1) N(d_-^{(2)})$$

$$= \left[x N(d_+^{(1)}) - K_1 e^{-r\tau} N(d_-^{(1)})\right] - \left[x N(d_+^{(2)}) - K_2 e^{-r\tau} N(d_-^{(2)})\right]$$

Thm 8.4 $V_T = g(S_T)$ Assume $f = f(t,x)$ satisfies BS PDE.

Then this option is replicable and its replicating port $f(t, S_t)$

pf $\left.\begin{cases} X_0 = f(0, S_0) \\ \Delta_t = \partial_x f(t, S_t) \end{cases}\right\} \rightarrow \exists$ self-fin X

WTS X is a replicating port & $X_t = f(t, S_t)$

$$\begin{cases} X \text{ : self-fin } \checkmark \\ X_T = V_T \ \checkmark \end{cases}$$

It's sufficient to prove $X_t = f(t, S_t)$

$\because$ $X_T = f(T, S_T) = g(S_T) = V_T$

In class, $dX_t = df(t, S_t)$.

$$d e^{-rt} X_t = d e^{-rt} f(t, S_t)$$

$$\cdot\quad d e^{-rt} X_t = -r\cdot e^{-rt} X_t \, dt + e^{-rt} dX_t + 0$$

$$= -r e^{-rt} X_t \, dt + e^{-rt} \left(\Delta_t \, dS_t + r(X_t - \Delta_t S_t) dt \right)$$

$$= (-r \Delta_t e^{-rt} S_t) \, dt + e^{-rt} \Delta_t \, dS_t .$$

$$\cdot\quad d e^{-rt} f(t, S_t) = (-r e^{-rt} f + e^{-rt} \cdot \partial_t f) \, dt + e^{-rt} \partial_x f \, dS_t$$

$$+ \frac{1}{2} e^{-rt} \partial_x^2 f \, \underbrace{d\langle S, S\rangle_t}_{= \sigma^2 S_t^2 dt}$$

$$= e^{-rt} \left(-rf + \partial_t f + \frac{1}{2} \partial_x^2 f \cdot \sigma^2 S_t^2 \right) dt + e^{-rt} \partial_x f \, dS_t$$

$$\partial_t f + rx \partial_x f + \frac{1}{2} \sigma^2 x^2 \partial_x^2 f = rf \Rightarrow -rf + \partial_t f + \frac{1}{2} \sigma^2 x^2 \partial_x^2 f = -rx \partial_x f$$

$\blacksquare$