$\boxed{Q2a}$ $E_t f(W_T)$ $\underset{\text{know (HW4)}}{=}$ $\varphi(t, W_t)$

Have a formula for φ = ______

Show $\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = 0$

$\boxed{\text{Method 1:}}$ Find formula for φ & diff.

$\varphi(t, W_t) = E_t f(W_T) = E_t f(W_T - W_t + W_t)$

Indep lemma $g(t, W_t)$ $(g = \varphi)$

Where $g(t,x) = E\, f(W_T - W_t + x)$

$$= \int_{-\infty}^{\infty} f(y+x) \frac{e^{-y^2/2(T-t)}}{\sqrt{2\pi(T-t)}}\, dy$$

$z = y + x$

$$= \underbrace{\int_{-\infty}^{\infty} f(z) \frac{e^{-(z-x)^2/2\tau}}{\sqrt{2\pi\tau}}}_{\varphi(t,x)} \qquad \tau = T - t$$

Compute $\partial_x^2 \varphi = \partial_x^2 \int_{-\infty}^{\infty} f(z) \frac{e^{-(z-x)^2/2\tau}}{\sqrt{2\pi\tau}}$

$$= \int_{-\infty}^{\infty} f(z) \; \partial_x^2 \left(\quad \right)$$

$$\rightarrow \partial_t \varphi = \partial_t \int_{-\infty}^{\infty} f(z) \frac{e^{-(z-x)^2/2\tau}}{\sqrt{2\pi\tau}} \, dz$$

$$= \int_{-\infty}^{\infty} f(z) \; \partial_t \left(\quad \right) dz$$

Write $\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi$ & check $= 0$

Method 2

I claim there is a much shorter & easier method using Itô formula

Q1 (b) $M_t = \int_0^t W_s \, dW_s$.

Compute $E\left((M_t - M_s)^2 W_s^2\right) = E\left[\overbrace{E_s (M_t - M_s)^2 W_s^2}\right]$

$= E\left(W_s^2 \; E_{\underline{s}} (M_t - M_s)^2\right) \overset{\text{Ito Isom}}{=}$

Recall: Ito Isom: $E_{\underline{0}}\left(\int_0^t D_r \, dW_r\right)^2 = E_{\underline{0}} \int_0^t D_r^2 \, dr$

But $E_s\left(\int_0^t D_r \, dW_r\right)^2 \neq E_s \int_0^t D_r^2 \, dr$

$\boxed{\text{Isom works if } \quad E_s\left(\int_s^t D_r\, dW_r\right)^2 = E_s\int_s^t D_r^2\, dr}$ True

$\Rightarrow E\left(W_s^2 \; E_s (M_t - M_s)^2\right)$

$= E\left(W_s^2 \; E_s\left(\int_s^t W_r\, dW_r\right)^2\right)$

$\underset{\text{Ito Isom}}{=} E\left(W_s^2 \; E_s \int_s^t W_r^2\, dr\right)$

$$= E\left(W_s^2 \int_s^t E_s W_r^2 \, dr\right)$$

$$= E\left(W_s^2 \int_s^t (W_s^2 - s + r) \, dr\right) \text{ \& evaluate.}$$

$$= E\left[W_s^2 (t-s)(W_s^2 - s) + W_s^2 \left[\frac{r^2}{2}\right]_s^t \right]$$

Q2b) Want $f(W_T) = E f(W_T) + \underline{\int_0^T \partial_x \varphi(s, W_s) \, dW_s}$.

Know $\overbrace{\underbrace{\varphi(t,W_t) = E_t f(W_T)}}$ & $\underbrace{\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = 0}_{(a)}$

$$\Rightarrow \underbrace{\varphi(T,W_T) = E_T f(W_T) = f(W_T).}$$

Itô : $d\varphi(t,W_t) = \partial_t \varphi \, dt + \partial_x \varphi \, dW_t + \frac{1}{2}\partial_x^2 \varphi \, d\overbrace{[W,W]}^{dt}$

$$= \Big(\underbrace{\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi}_{0}\Big) dt + \partial_x \varphi \, dW_t$$

$$\Rightarrow \overbrace{\varphi(T,W_T)}^{f(W_T)} - \underbrace{\varphi(0,W_0)} = \int_0^T \underline{\underline{\partial_x \varphi(s,W_s)\, dW_s}}$$

$$u(0, W_0) = E_0 f(W_T) = E f(W_T)$$