Midterm: Went well !!

8. Black Scholes Merton equation

8.1. Market setup and assumptions.

- Cash: simple interest rate r in a bank.
- Let Δt be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$.
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t)C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\partial_t C = rC$ and $C_t = C_0 e^{rt}$.
- r is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $r = \frac{1}{T}\ln(1+\rho)$.

$$\boxed{t = n\Delta t}$$

$$C_{(n+1)\Delta t} = (1 + r\Delta t)\, C_{n\Delta t}$$

$$\iff \frac{C_{(n+1)\Delta t} - C_{n\Delta t}}{\Delta t} = r\, C_{n\Delta t}$$

$$\xrightarrow{\Delta t \to 0} \quad \partial_t C_t = r\, C_t.$$

$$(1+\rho) = e^{rT}$$

$$\iff r = \frac{1}{T}\ln(1+\rho)$$

- Stock price: $S_{t+\Delta t} = (1 + \overset{\alpha}{\cancel{\gamma}}\Delta t)S_t + \text{noise}$.
  - ▷ Variance of noise should be proportional to $\boxed{\Delta t}$.
  - ▷ Variance of noise should be proportional to $\boxed{S_t}$?
- $S_{t+\Delta t} - S_t = \underset{\alpha}{\cancel{\gamma}} S_t\,\Delta t + \sigma S_t(\Delta W_t)$.

$$S_{t+\Delta t} = (1 + \alpha \Delta t) S_t + \underbrace{\sigma S_t \boxed{\Delta W_t}}_{\text{noise}}$$

GBM

Definition 8.1. A Geometric Brownian motion with parameters α, σ is defined by:

$$\boxed{dS_t = \alpha S_t\, dt + \sigma S_t\, dW_t\,.}$$

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Typical first model for stock price.

Proposition 8.2. $\boxed{S_t = S_0 \exp\Big(\Big(\alpha - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)}$

Compute:
$$dS_t = \alpha S_t\, dt + \sigma S_t\, dW_t$$

$$\Rightarrow \quad \frac{dS_t}{S_t} = \alpha\, dt + \sigma\, dW_t$$

$$f(x) = \ln x$$
$$\partial_t f = 0$$
$$\partial_x f = \frac{1}{x}$$
$$\partial_x^2 f = -\frac{1}{x^2}$$

Apply Itô to $\ln(S_t)$:

$$d(\ln S_t) = 0 + \frac{1}{S_t}\, dS_t - \frac{1}{2}\frac{1}{S_t^2}\, \underbrace{d[S,S]_t}_{\sigma^2 S_t^2\, dt}$$

$$\Rightarrow \quad d(\ln S_t) = \frac{1}{S_t}\left(\alpha S_t\, dt + \sigma S_t\, dW_t\right) - \frac{1}{2S_t^2}\sigma^2 S_t^2\, dt$$

$$= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

Integrate $\Rightarrow \underbrace{\ln S_t - \ln S_0}_{\ln\left(\frac{S_t}{S_0}\right)} = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t$

$$\boxed{\Rightarrow S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$$

Market Assumptions.

- 1 stock, Price S_t, modelled by GBM(α, σ).
- Money market: Continuously compounded interest rate $\boxed{r}$.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = rC_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)

GBM(α, r)

Stock

Bank

Market

(cts time)

8.2. **The Black, Sholes, Merton equation.** Consider a security that pays $\boxed{V_T = g(S_T)}$ at maturity time T.

Theorem 8.3. *If the security can be replicated, and $f = f(t,x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{8.1}$$

(B.S. PDE).

$$f(t,0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{8.2}$$

(Bounding cond).

$$\boxed{f(T,x) = g(x)} \qquad x \geqslant 0\,. \tag{8.3}$$

Theorem 8.4. *Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 8.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 8.6. $g(x) = (x - K)^+$ is a European call with strike K and maturity T.

Remark 8.7. $g(x) = (K - x)^+$ is a European put with strike K and maturity T.

At maturity

$X_T = f(T, \cancel{S_T}^{\,x})$

$= g(\cancel{S_T}^{\,x})$

(8.2): If $S_t = 0$ then for all later times $S_t = 0$

$\Rightarrow S_T = 0 \Rightarrow$ Payoff $V_T = g(0)$

AFP at time $t = g(0)\, e^{-r(T-t)}$ (discounting). $\to$ 8.2

Proposition 8.8. *A standard change of variables gives an explicit solution to* (8.1)–(8.3)*:*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}\,, \qquad \boxed{\tau = T - t\,.} \tag{8.4}$$

Corollary 8.9. *For European calls, $g(x) = (x-K)^+$, and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

is the CDF of a standard normal variable.

Will derive (8.4) & (8.5) using RNM in last lecture.

Remark 8.10. Equation (8.1) is called a $\boxed{\textit{partial differential equation.}}$ In order to have a unique solution it needs:

(1) A terminal condition (this is equation (8.3)),
(2) A boundary condition at $x = 0$ (this is equation (8.2)),
(3) A boundary condition at infinity (not discussed yet).

▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is

$$\lim_{x \to \infty} f(t, x) = 0\,.$$

▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is

$$\lim_{x \to \infty} \left[f(t, x) - (x - Ke^{-r(T-t)}) \right] = 0 \qquad \text{or} \qquad f(t, x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty\,.$$

Definition 8.11. If X_t is the wealth of a self-financing portfolio then

$$\boxed{dX_t = \Delta_t\, dS_t + r(X_t - \Delta_t S_t)\, dt}$$

for some adapted process Δ_t (called the trading strategy).

Disc time: Self fin means

$$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r)$$

$$\iff X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n)$$

$$\boxed{dX_t = \Delta_t\, dS_t + r(X - \Delta_t S_t)\boxed{dt}}$$

Proof of Theorem 8.3. If $X_t = f(t, S_t)$ = wealth of R. Port

then f satisfies BS PDE 8.1 8.2 8.3

$$\boxed{\partial_t f + r x \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f = r f}$$

$$\begin{cases} dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t \\ d[S,S] = \sigma^2 S_t^2 \, dt \end{cases}$$

1. Know $X_t = f(t, S_t)$

Compute dX_t using Itô

$$d f(t, S_t) = \partial_t f \, dt + \partial_x f \, dS_t + \frac{1}{2} \partial_x^2 f \, d[S,S].$$

$$\Rightarrow \quad d f(t, S_t) = \partial_t f \, dt + \partial_x f \left(\alpha S_t \, dt + \sigma S_t \, dW_t \right) + \frac{1}{2} \partial_x^2 f \, \sigma^2 S_t^2 \, dt$$

$$\boxed{d f(t, S_t) = \left(\partial_t f + \alpha \partial_x f \, S_t + \frac{\sigma^2}{2} \partial_x^2 f \, S_t^2 \right) dt + \left(\sigma \partial_x f \, S_t \right) dW_t} \quad (*)$$

(2) Know X is self fin

$$\Rightarrow \quad dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t) \, dt$$

$$= \Delta_t \left(\alpha S_t \, dt + \sigma S_t \, dW_t \right) + r(X_t - \Delta_t S_t) \, dt$$

$$\boxed{\Rightarrow dX_t = \left(rX_t + (\alpha - r)\Delta_t S_t\right)dt + \sigma \Delta_t S_t \, dW_t \quad (**)}$$

Δ_t = # shares at time t in Rep port.

(3) Equate (*) & (**)

Uniq of Itô $\Rightarrow$ the dt terms must be equal
& the dW " " " "

Equate dW terms: $\Rightarrow \cancel{\sigma}\Delta_t \cancel{S_t} = \cancel{\sigma}\,\partial_x f \,\cancel{S_t}$

$$\boxed{\Rightarrow \Delta_t = \partial_x f(t, S_t)} \leftarrow \text{Delta Hedging Rule.}$$

Equate dt terms:

$$\Rightarrow r X_t + (\alpha - r)\Delta_t S_t = \partial_t f + \alpha S_t \partial_x f + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

Substitute $X_t = f(t, S_t)$ & $\Delta_t = \partial_x f$

$$\Rightarrow r f + (\cancel{\alpha} - r) S_t \partial_x f = \partial_t f + \cancel{\alpha S_t \partial_x f} + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

$$\Rightarrow r f = \partial_t f + r S \partial_x f + \frac{\sigma^2}{2} S_t^2 \partial_x^2 f$$

Write x instead of S_t get B.S. PDE (8.1) !

Proof of Theorem 8.4. Assume f solves BS PDE

Want to show $f(t, S_t) =$ AFP $=$ Wealth of R. Port.

1. Let $Y_t = f(t, S_t)$

2. Let $X_t =$ wealth of a self fin port

   Choose $X_0 = f(0, S_0) = Y_0$

   Choose $\Delta_t = \#$ shares held at time t

   $= \partial_x f(t, S_t)$ (Delta Hedging)

Claim: X is the Rep Port.

i.e. $X_T = g(S_T) \overset{(8.3)}{=} f(T, S_T) = Y_T$.

Will show: For all $t \geq 0$, $X_t = Y_t = f(t, S_t)$.

① Compute dX_t:

$$dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\, dt$$

$$= \Delta_t(\alpha S_t \, dt + \sigma S_t \, dW_t) + r(X - \Delta S)\, dt$$

$$\boxed{dX_t = \left((\alpha - r)\Delta_t S_t + r X_t \right) dt + \sigma \Delta_t S_t \, dW_t}$$

(***)

② Compute dY: $\quad Y = f(t, S_t)$

$$\Rightarrow dY = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2} \partial_x^2 f \, d[S, S]$$

$$= \partial_t f \, dt + \partial_x f \left(\alpha S \, dt + \sigma S \, dW \right) + \frac{1}{2} \partial_x^2 f \, \sigma^2 S^2 \, dt$$

$$dY_t = \left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2}{2} S^2 \partial_x^2 f\right) dt + \sigma \partial_x f \, S \, dW$$

$$\Rightarrow d(Y - X) = \left(\qquad \right) dt + \left(\sigma \partial_x f \, S - \sigma \Delta_t S\right) dW \longrightarrow 0$$

$$= \left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2} \partial_x^2 f - (\alpha - r) \Delta_t S_t - r X_t \right) dt + 0 \, dW$$