Midtennie West well

## 8. Black Scholes Merton equation

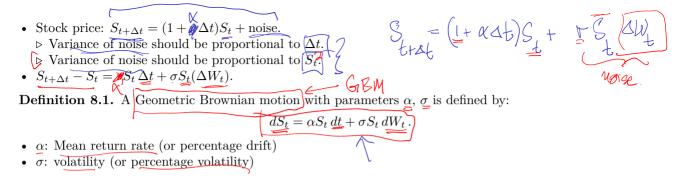
## 8.1. Market setup and assumptions.

- Cash: simple interest rate r in a bank.
- Let  $\Delta t$  be small.  $C_{n \Delta t}$  be cash in bank at time  $n \Delta t$ .
- Withdraw at time  $n \Delta t$  and immediately re-deposit:  $C_{(n+1)\Delta t} = (1 + r \Delta t) C_{n\Delta t}$ .
- Set  $t = n\Delta t$ , send  $\Delta t \to 0$ :  $\partial_t C = rC$  and  $C_t = C_0 e^{rt}$
- $\underline{r}$  is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate  $\rho$  after time T, then the equivalent continuously compounded interest rate is  $r = \frac{1}{T} \ln(1 + \rho)$ .

 $(1 + \rho)$ 

 $(=) r = \frac{1}{r} lu(1+p)$ 

Mab) MSL = 1



Typical finet model for stock price.

 $= d(t_{W} S_{t}) = \frac{1}{S_{L}} \left( x S_{t} dt + \tau S_{t} dW_{t} \right) - \frac{1}{2\varsigma_{t}^{2}} \tau^{2} \varepsilon_{t}^{2}$ 

 $=(\alpha - \overline{\alpha}^2) qt + \sigma dW_t$ 

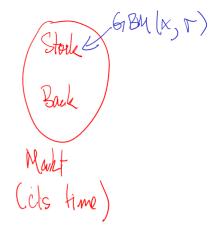
Intervie  $\rightarrow \ln S_{t} - \ln S_{t} = (\kappa - \tilde{\Sigma})t + \tau W_{t}$  $\Rightarrow S_{t} = S_{0} eab\left(\left(\alpha - \frac{\sigma^{2}}{2}\right)t + \sigma W_{t}\right)$  $\operatorname{Lu}\left(\frac{S_{\mathsf{L}}}{\overline{S}_{\mathsf{O}}}\right)$ 

## Market Assumptions.

- 1 stock, Price  $S_t$ , modelled by  $\text{GBM}(\alpha, \sigma)$ .
- Money market: Continuously compounded interest rate r.

   *C<sub>t</sub>* = cash at time t = C<sub>0</sub>e<sup>rt</sup>. (Or ∂<sub>t</sub>C<sub>t</sub> = rC<sub>t</sub>.)

   Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)



8.2. The Black, Sholes, Merton equation. Consider a security that pays  $V_T = g(S_T)$  at maturity time T. **Theorem 8.3.** If the security can be replicated, and f = f(t, x) is a function such that the wealth of the replicating portfolio is given by  $X_t = f(t, S_t)$ , then:  $\ni \underbrace{\partial_t f}_{\underline{x}} + \underbrace{rx \partial_x f}_{\underline{x}} + \frac{\sigma^2 x^2}{2} \underbrace{\partial_x^2 f}_{\underline{x}} - \underline{rf} = 0 \qquad x > 0, \ t < T, \qquad (B \leq P)E).$ (8.1) $f(t,0) = \left(g(0)e^{-r(T-t)} \quad t \leq T, \quad (\text{Bound ig cond})\right).$ (8.2) $f(\underline{T},\underline{x}) = g(\underline{x}) \qquad \qquad x \ge 0 \,.$ (8.3)**Theorem 8.4.** Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and  $X_t = f(t, S_t)$  is the wealth of the replicating portfolio at any time  $t \leq T$ . *Remark* 8.5. Wealth of replicating portfolio equals the arbitrage free price. Remark 8.6.  $g(x) = (x - K)^+$  is a European call with strike K and maturity T. Remark 8.7.  $g(x) = (K - x)^+$  is a European put with strike K and maturity T. (802): If S = O thin for all later time St = O  $\Rightarrow S_{T} = 0 \implies Payoff \quad V_{T} = g(0)$ AFP at time  $t = g(0) e^{-\gamma(T-t)}$ 

**Proposition 8.8.** A standard change of variables gives an explicit solution to (8.1)–(8.3):

(8.4) 
$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\left(x \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y\right)\right) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}, \quad \left(\tau = T - t\right)$$

**Corollary 8.9.** For European calls,  $g(x) = (x - K)^+$ , and

(8.5) 
$$f(t,x) = c(t,x) = xN(d_{+}(T-t,x)) - Ke^{-r(T-t)}N(d_{-}(T-t,x))$$

where

(8.6) 
$$d_{\pm}(\tau, x) \stackrel{\text{\tiny def}}{=} \frac{1}{\sigma\sqrt{\tau}} \left( \ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right) \tau \right),$$

and

(8.7) 
$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \, ,$$

is the CDF of a standard normal variable.

Remark 8.10. Equation (8.1) is called a *partial differential equation*. In order to have a unique solution it needs:

- (1) A terminal condition (this is equation (8.3)),
- (2) A boundary condition at x = 0 (this is equation (8.2)),
- (3) A boundary condition at infinity (not discussed yet).

▷ For put options,  $g(x) = (K - x)^+$ , the boundary condition at infinity is

$$\lim_{x \to \infty} f(t, x) = 0.$$

0

 $\triangleright$  For call options,  $g(x) = (x - K)^+$ , the boundary condition at infinity is

 $\lim_{x \to \infty} \left[ f(t,x) - (x - Ke^{-r(T-t)}) \right] = 0 \quad \text{or} \quad f(t,x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty \,.$ 

**Definition 8.11.** If  $X_t$  is the wealth of a self-financing portfolio then

$$dX_t = \underbrace{\Delta_t}_{\cong} \underbrace{dS_t}_{t} + \underbrace{r}(\underbrace{X_t - \Delta_t S_t}_{t}) dt$$

for some adapted process  $\Delta_t$  (called the trading strategy).

Vice time: Self fin nears  $X_{\mu\mu} = \Delta_{n} S_{\mu\mu} + (X_{n} - \Delta_{n} S_{n}) (I+r)$  $\Rightarrow X_{nH} - X_{n} = \Delta_{n} (S_{nH} - S_{n}) + \chi (X_{n} - \Delta_{n} S_{n})$  $dX_t = \Delta_t dS_t + \Upsilon(X - \Delta_t S_t) dt$ 

Proof of Theorem 8.3. If  $X_{t} = f(t_{2}S_{t}) = wealth, R. Point$ thun & satisfier BS PDE (8.1) (8.2) (8.3)  $\left|\partial_{t}\right| + \gamma \chi_{t}^{2} + \frac{\Gamma^{2} \chi^{2}}{2} \partial_{x}^{2} = \gamma_{t}^{2}$  $dS_t = \alpha S_t dt + \tau S_t dW_t$   $dS_t = \tau^2 S_t^2 dt$ (1) Know  $X_{+} = f(t, S_{+})$ Comparte d'X, acing Ito  $\partial_{xb} dS_{t} + \frac{1}{2} \partial_{x}^{2} f d(S, S)$  $d(f(t,S_t) = \partial_{th} dt +$ 

 $\Rightarrow d \left[ (t_j S_j) = \partial_j f_j db + \partial_x \right] \left( \alpha S_j dt + \sigma S_j dW_j \right) + \frac{1}{2} \partial_x^2 \Big[ \sigma^2 S_j^2 dt \right]$  $d\left[\left(6,5\right) = \left(2,1\right) + \left(3,1\right) + \left$ E Know X is self for  $\Rightarrow dX_t = \Delta_t dS_t + \gamma(X_t - \zeta_t S_t) dt$  $= \Delta_{t} \left( x S_{t} dt + \sigma S_{t} dW_{t} \right) + \gamma \left( X_{t} - A_{t} S_{t} \right) dt$ 

$$= dX_{t} = (rX_{t} + (r-r)A_{t}S_{t})dt + rA_{t}S_{t}dW_{t}$$

$$= (rX_{t} + (rX_{t} + (r)A_{t})dt + rA_{t}A_{t}dW_{t}$$

$$= (rX_{t} + (rX_{t} + (r)A_{t})dt + rA_{t}A_{t}dW_{t}$$

$$= (rX_{t} + ($$

at = 2 f(t, St) (= Dolta Hedging Rule. Equate de tems ?  $\Rightarrow \tau X_{t} + (\alpha - \tau) \Delta_{t}^{2} S_{t} = \partial_{t} \delta_{t} + \alpha S_{t} \partial_{x} \delta_{t} + \frac{\tau}{2} S_{t}^{2} \delta_{x} \delta_{t}$  $\Rightarrow x_{b} + (\alpha - v) S_{t} \partial_{x_{b}} = \partial_{t_{b}} + \alpha S_{t} \partial_{x_{b}} + \frac{v^{2}}{2} S_{t}^{2} \partial_{x_{b}} f$ 

1

 $\Rightarrow \gamma_{\downarrow} = \partial_{\downarrow} \xi + \gamma S \partial_{\chi} \xi + \frac{\nabla^2 S_{\downarrow}^2}{2} \partial_{\chi}^2 \xi$ Ware x instead of St get B.S. PDE (Bil)

Proof of Theorem 8.4. Alsung & Galves BS PDE Want to show  $f(t, S_t) = AFP = Weath of R. Port.$ () but  $Y_{t} = f(t, S_{t})$ (2) het X = wealth of a self for pourt Choose  $X_n = \{(0, S_n) = Y_n\}$ Choose a = # shoes held at time t = Z\_{f(t, S\_{f})} (Delta Honging)

Claimin X is the Rep Pant. i.e  $X_T = g(S_T) = f(T, S_T) - Y_T$ Will choos: For all  $t \ge 0$ ,  $X_t = Y_t = \left\{ (t_3 S_t) \right\}$ . () Compute d X?  $dX_{t} = \Delta_{t} dS_{t} + \gamma (X_{t} - \Delta_{t} S_{t}) dt$  $= \Delta_t \left( \alpha S_t dt + \tau S_t dW_t \right) + \tau \left( X - \alpha S \right) dt$ 

$$dX_{t} = \left((k-r)\Delta_{t}S_{t} + rX_{t}\right)dt + r\Delta_{t}S_{t}dW_{t}$$

$$3 \lim_{x \to 0} \frac{1}{2} \int \frac{1}{2} \frac{1}{2} \int \frac{1}$$

 $dY_{t} = \left(\partial_{t}f + \alpha S\partial_{x}f + \frac{\pi^{2}}{2}S^{2}\partial_{x}f\right)dt + \nabla\partial_{x}f SdW$ 

 $\exists d(Y-X) = () dt + (rdS-rdS)dW$ 

 $= \left( \frac{\partial_{\xi} b}{\partial t} + \alpha S \partial_{\chi} b + \frac{\tau^2 S^2}{2} \partial_{\chi}^2 \right) + D dW.$ -  $(\alpha - \gamma) \mathcal{A} S_{\xi} - \gamma X_{\xi} dt$