$$X_t = \int_0^t \sigma_s \, dW_s \to \text{mart}$$

$$\left.\begin{aligned} Y_t &= \int_0^t b_s \, dX_s \\ &= \int_0^t b_s \sigma_s \, dW_s \end{aligned}\right] \to \text{mart}$$

HW4 Q.5

$$\left[\begin{aligned} &\mathbb{P}(X=2) = 0.3 \\ &\mathbb{P}(X=3) = 0.7 \\ &A = \{\mathbb{E}_u[X] = \alpha\} \in \mathcal{F}_u \ ? \\ &\mathbb{P}(A) = 0.6 \\ &\mathbb{P}(A \cap \{X=2\}) = 0.2 \end{aligned}\right.$$

<u>Def</u> Y is condi exp of X at time u

① Y is $\mathcal{F}_u$-m'ble

② $\forall B \in \mathcal{F}_u$, $\mathbb{E}[Y \cdot \mathbb{1}_B] = \mathbb{E}[X \cdot \mathbb{1}_B]$

$$\mathbb{E}\left[\mathbb{E}_n[X]\cdot \mathbb{1}_A\right] = \mathbb{E}\left[X\cdot \mathbb{1}_A\right]$$

$$\mathbb{E}\left[\mathbb{E}_n[X]\cdot \mathbb{1}_A\right] = \alpha\cdot \mathbb{P}(A) = \alpha\cdot 0.6$$

$$\Rightarrow \alpha = \ldots$$

$$\begin{aligned}
\mathbb{E}\left[X\cdot \mathbb{1}_A\right] &= \mathbb{E}\left[X\cdot \mathbb{1}_{(\{X=2\}\cap A)}\right] + \mathbb{E}\left[X\cdot \mathbb{1}_{(\{X=3\}\cap A)}\right] \\
&= 2\cdot \mathbb{P}(\{X=2\}\cap A) + 3\cdot \mathbb{P}(\{X=3\}\cap A) \\
&= 2\cdot 0.2 + 3\cdot\left(\mathbb{P}(A) - \mathbb{P}(\{X=2\}\cap A)\right) \\
&= 2\cdot 0.2 + 3(0.6 - 0.2)
\end{aligned}$$

2020. #4 $\quad \mathbb{E}\left[e^{2W_sW_t} \mid \mathcal{F}_s\right] \quad s<t$

$$\mathbb{E}\left[e^{2W_sW_t} \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{2W_s(W_t-W_s)}\cdot e^{2W_s^2} \mid \mathcal{F}_s\right]$$

$$= e^{2W_s^2}\,\mathbb{E}\left[e^{2W_s(W_t-W_s)} \mid \mathcal{F}_s\right]$$

$$= e^{2W_s^2} g(W_s)$$

where $g(x) = \mathbb{E}\left[e^{2x \overbrace{(W_t - W_s)}^{} } \right]$ $\sim N(0, t-s)$

$$= e^{2x^2(t-s)}$$

2019. #5

$$\mathbb{E}_s\left[\int_s^t 2M_r\,dM_r\right] = 0.$$

$X_t = \int_0^t 2M_r\,dM_r \rightarrow$ martingale

$$\int_s^t 2M_r\,dM_r = X_t - X_s$$

$$\mathbb{E}_s[X_t - X_s] = \mathbb{E}_s[X_t] - X_s$$
$$= X_s - X_s = 0$$

2021 #5 $\quad M_t = \int_0^t s W_s\,ds \quad dM_t = tW_t\,dt$

$$dM_t^2 = 2M_t\,dM_t + \cancelto{0}{d\langle M,M\rangle_t}$$

$$\Rightarrow M_t^2 = \int_0^t 2M_s\,dM_s$$

$$= \int_0^t 2\cdot M_s\cdot s\cdot W_s\,ds$$

$$\Rightarrow \mathbb{E}[M_t^2] = \int_0^t 2\cdot s\;\mathbb{E}[M_s\cdot W_s]\,ds$$

$$\mathbb{E}[M_s\cdot W_s] = \mathbb{E}\left[W_s\cdot\int_0^s r W_r\,dr\right]$$

$$= \mathbb{E}\left[\int_0^s W_s\cdot r W_r\,dr\right]$$

$$= \int_0^s r\cdot \underbrace{\mathbb{E}[W_s\cdot W_r]}_{s\wedge r}\,dr$$

$$= \int_0^s r^2\,dr = \frac{s^3}{3}$$

$$\Rightarrow \mathbb{E}[M_t^2] = \int_0^t \frac{2}{3}\cdot s^4\,ds = \frac{2t^5}{15}$$

#7.6 $M_t = \int_0^t \sigma_s \, dW_s$. σ: deterministic

(1)

$$\underbrace{E[e^{\lambda M_t}]}_{\text{mgf of Gaussian}}, \quad \underline{E_s[e^{\lambda(M_t - M_s)}]}$$

$$\underline{X_t} = e^{\lambda(M_t - M_s)} = e^{\lambda \int_s^t \sigma_r dW_r}$$

$$\underline{X_s = 1} \qquad\qquad = \underline{f(M_t)}$$

where $M_t = \int_s^t \sigma_r dW_r$, $f(x) = e^{\lambda x}$

$\partial_x f = \lambda \cdot f$

$\partial_x^2 f = \lambda^2 \cdot f$

$$dX_t = \lambda \cdot f \, dM_t + \frac{1}{2}\lambda^2 f \, d\langle M, M\rangle_t$$

$$= \lambda X_t \sigma_t dW_t + \frac{1}{2}\lambda^2 X_t \sigma_t^2 dt$$

$$\Rightarrow X_t = 1 + \underline{\int_s^t \lambda X_r \sigma_r dW_r} + \frac{\lambda^2}{2}\int_s^t X_r \sigma_r^2 dr$$

$$\Rightarrow \underbrace{\mathbb{E}_s[X_t]}_{\varphi(t)} = 1 + 0 + \mathbb{E}_s\left[\frac{\lambda^2}{2}\int_s^t X_r \sigma_r^2 \, dr\right]$$

$$= 1 + \frac{\lambda^2}{2}\int_s^t \underbrace{\mathbb{E}_s[X_r]}_{\varphi(r)} \sigma_r^2 \, dr$$

$$\varphi'(t) = \frac{\lambda^2}{2}\cdot\varphi(t)\cdot\sigma_t^2, \qquad \varphi(s) = X_s$$

$$\Rightarrow \varphi(t) = X_s \cdot e^{\frac{\lambda^2}{2}\int_s^t \sigma_r^2 \, dr} \quad \checkmark$$

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(2) $$\mathbb{E}\left[e^{\lambda M_r + \mu(M_t - M_s)}\right]$$ = mgf of $(M_r, M_t - M_s)$

$$= \mathbb{E}\left[\mathbb{E}_s\left[e^{\lambda M_r + \mu(M_t - M_s)}\right]\right]$$

$$= \mathbb{E}\left[e^{\lambda M_r} \cdot \mathbb{E}_s\left[e^{\mu(M_t - M_s)}\right]\right]$$

$$= \mathbb{E}\left[e^{\lambda M_r} \cdot \underbrace{X_s}_{1} \cdot e^{\frac{\mu^2}{2}\int_s^t \sigma_u^2 \, du}\right]$$

$$= \underline{\mathbb{E}\left[e^{\lambda M_r}\right]} e^{\frac{\mu^2}{2}\int_s^t \sigma_u^2 du}$$

$$= e^{\frac{\lambda^2}{2}\int_0^r \sigma_u^2 du} \cdot e^{\frac{\mu^2}{2}\int_s^t \sigma_u^2 du}$$

= mgf of $\underline{(X, Y)}$ where

$\left(\underline{X \perp\!\!\!\perp Y} \quad \begin{array}{l} \checkmark X \sim N(0, \int_0^t \sigma_u^2 du) \\ \checkmark Y \sim N(0, \int_s^t \sigma_u^2 du) \end{array} \right)$

HW4 #4 $\quad \varphi(t, W_t) = \underbrace{\mathbb{E}_t[f(W_T)]}$

$\rightsquigarrow \quad \varphi(t, W_t)$ is a mart.

$$d\varphi(t, W_t) = (\underbrace{\partial_t \varphi + \tfrac{1}{2}\partial_x^2 \varphi}_{=0})\,dt + \partial_x \varphi\, dW_t$$

$$\Rightarrow d\varphi(t, W_t) = \partial_x \varphi\, dW_t$$

$$\Rightarrow \quad \underbrace{\varphi(T, W_T)}_{\substack{\shortparallel \\ \mathbb{E}_T[f(W_T)] \\ \shortparallel \\ f(W_T)}} = \underbrace{\varphi(0, W_0)}_{\substack{\shortparallel \\ \mathbb{E}[f(W_T)]}} + \int_0^T \partial_x \varphi(s, W_s)\, dW_s$$

2019 #5.

$$\underline{\mathbb{E}_s[M_t^2]}$$

$$= \mathbb{E}_s[M_t^2 - \langle M, M\rangle_t] + \mathbb{E}_s[\langle M, M\rangle_t]$$

$$= \underline{M_s^2 - \langle M, M\rangle_s} + \underline{\mathbb{E}_s[\langle M, M\rangle_t]}$$