Q 7.6 $\qquad M \to M_g \qquad d[M,M]_t = \sigma_t$ (σ not random)

(i) $\underbrace{E\, e^{\lambda M_t}}$ (Compute)

MGF of M_t.

$$f(t,x) = e^{\lambda x}$$

Let $\boxed{\varphi_t = E\, e^{\lambda M_t}}$

Ito on $e^{\lambda M_t}$: $\quad d(e^{\lambda M_t}) = \partial_t f\, dt + \partial_x f\, dM_t + \frac{1}{2}\partial_x^2 f\, d[M,M]$

$$= 0 + \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} e^{\lambda M_t} \sigma_t^2 \, dt$$

Integrate : $\underbrace{E e^{\lambda M_t}}_{\varphi_t} - \cancel{e^{\lambda M_0}}^{1} = E \lambda \int_0^t e^{\lambda M_s} dM_t + \underbrace{\frac{\lambda^2}{2} E \int_0^t e^{\lambda M_s} \sigma_s^2 \, ds}_{\text{R-int}}$

Expected val

$E \lambda \int_0^t e^{\lambda M_s} dM_t = 0$ ($\because$ M is a mg $\Rightarrow \int_0^t D_s \, dM_s$ is a mg)

$\frac{\lambda^2}{2} E \int_0^t e^{\lambda M_s} \sigma_s^2 \, ds = \frac{\lambda^2}{2} \int_0^t \underbrace{E e^{\lambda M_s}}_{\varphi_s} \, \sigma_s^2 \, ds$

not random

$$\Rightarrow \varphi_t - 1 = 0 + \frac{\lambda^2}{2} \int_0^t \varphi_s \sigma_s^2 \, ds$$

$$\Rightarrow \frac{d\varphi}{dt} = \frac{\lambda^2}{2} \varphi_t \sigma_t^2$$

$$\Rightarrow \frac{d\varphi}{\varphi} = \frac{\lambda^2}{2} \sigma^2 \, dt \quad \Rightarrow \frac{d}{dt}(\ln \varphi) = \frac{\lambda^2 \sigma^2}{2}$$

$$d(\ln \varphi)$$

Int

$$\Rightarrow \ln \varphi_t - \ln \varphi_0 = \frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds$$

$\varphi_0 = 1$

$$\Rightarrow \ln \varphi_t = \frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds \quad (\because \ln 1 = 0)$$

$$\boxed{\Rightarrow E e^{\lambda M_t} = \varphi_t = \exp\left(\frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds\right)}$$

(Note Rest of this question is similar to the above & the logic we used to prove Lévy's thm in class)

Q 6.41 : $X_t = t \sin(W_t)$

Is $X^2 - [X,X]$ a mg?

① $Y_t = X_t^2 - [X,X]_t$. Compute $dY_t = 2X_t\, dX_t \ldots$ ⊛

(Set $f(t,x) = x^2 - [X,X]_t$ & apply Ito)

② Compute dX_t (Set $f(t,x) = t \sin x$ & apply Ito)

Let $$dX_t = \sin(W_t)\,dt + t\cos W_t\,dW_t - \frac{t}{2}\sin(W_t)\,dt$$

$$= \underbrace{\left(\sin(W_t) - \frac{t}{2}\sin(W_t)\right)}_{\neq 0} dt + t\cos W_t\,dW_t$$

$\Rightarrow X$ is not a mg

$\Rightarrow$ from $(*)$ Y is not a mg.

Check $dY_t = 2X_t\,dX_t$:

$$f(t,x) = x^2 - [X,X]_t$$

$$\partial_t f = -\frac{d}{dt}[X,X]_t$$

$$\partial_x f = 2x \qquad \partial_x^2 f = 2$$

$$dY = -\cancel{d[X,X]_t} + 2X_t\,dt + \cancel{d[X,X]_t} = dX_t\,dX_t.$$

[HW 4 Q1 d]

$$X_t = W_t \int_0^{W_t} \exp(-ts^2)\,ds. \quad \text{Find Ito decomp.}$$

Apply Itô: Choose $f(t,x) = x\int_0^x \exp(-ts^2)\,ds$

$$\partial_t f = \frac{d}{dt}\left(x \int_0^{x} e^{-ts^2}\, ds \right)$$

$$= x \int_0^x \frac{d}{dt} e^{-ts^2}\, ds = \boxed{x \int_0^x \left(-s^2 e^{-ts^2}\right) ds}$$

$$\boxed{\frac{d}{dt}\int_0^t \frac{1}{x}\, dx = \frac{1}{t}}$$

$$\partial_x f = \int_0^x e^{-ts^2}\, ds + x\, e^{-tx^2}$$

$$\partial_x^2 f = e^{-tx^2} + x\left(-t e^{-x^2}\right) + e^{-tx^2}$$

$$dX_t = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt$$

$$= \left(W_t \int_0^{W_t} -s^2 e^{-ts^2} \, ds \right) dt + \left(\quad \right) dW$$

$$+ \left(\quad \right) dt$$

combine.

HW4 Q2[b]

$$X_t = \left(W_t + \frac{t^2}{2}\right) \exp\left(-\underline{\int_0^t s\, dW_s} - \frac{t^3}{6}\right)$$

Is X a mg.

Idea: If, $X_t = f(t, W_t)$ then use Ito & check if dt terms are 0

Try $f(t,x) = \left(x + \underline{\underline{\frac{t^2}{2}}}\right) \exp\left(-\underline{\underline{\int_0^t s\, dW_s}} - \frac{t^3}{6}\right)$ Doesn't work because $\partial_t f$ DNE!!

Try #2: Can we compute $\boxed{\displaystyle\int_0^t s\,dW_s\ ?}$

Chose $f(t,x) = tx$. $\partial_t f = x$, $\partial_x f = t$ & $\partial_x^2 f = 0$

$$\Rightarrow d(tW_t) = W_t\,dt + t\,dW_t + 0$$

$$\Rightarrow tW_t - 0 = \int_0^t W_s\,ds + \boxed{\int_0^t s\,dW_s}$$

$$\Rightarrow \int_0^t s\,dW_s = -tW_t + \int_0^t W_s\,ds \leftarrow \text{R int (diff in time)}$$

$$\Rightarrow X_t = \left(W_t + \frac{t^2}{2}\right) \exp\left(-\int_0^t s\, dW_s - \frac{t^3}{6}\right)$$

$$= \left(W_t + \frac{t^2}{2}\right) \exp\left(-t W_t + \int_0^t W_s\, ds\right)$$

$$= g(t, W_t) \quad \text{where}$$

$$g(t,x) = \left(x + \frac{t^2}{2}\right) \exp\left(-tx + \int_0^t W_s\, ds - \frac{t^3}{6}\right)$$

To apply Itô we need

① $\partial_t g$ (Works because $\frac{d}{dt}\int_0^t W_s\,ds = W_t$)

② $\partial_x g$ & $\partial_x^2 g$ to all exist !

Compute $\partial_t g = (x+\frac{t^2}{2})\exp(\quad)\cdot\left(-x+W_t-\frac{t^2}{2}\right)$

$+ \exp(\quad)\cdot t$

& $\partial_x g = (x+\frac{t^2}{2})\exp(\quad)\cdot(-t) + \exp(\quad)\cdot 1$

$\partial_x^2 g = -t(x+\frac{t^2}{2})\exp(\,)\cdot(-t) + \exp(\,)(-t) + \exp(\,)(-t).$

& substitute into Itô.