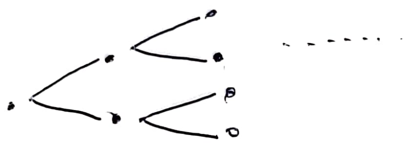
I. Discrete time case

$\Omega = \{N\text{-many coin tosses}\}$, p, q



1. Math Theory

<u>Def</u> (Filtration) Filtration: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ where

$\mathcal{F}_n = \{\text{events that only depend on the first } n\text{-coins}\}$

$\mathcal{F}_0 = \{\phi, \Omega\}$

<u>Def</u> (Measurability) A r.v $X : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-m'ble if

$\{X \in B\} \in \mathcal{F}_n, \quad \forall B \subseteq \mathbb{R}$

Def (Condi Exp) A r.v. $X: \Omega \to \mathbb{R}$. $\mathbb{E}_n[X] = \mathbb{E}[X \mid \mathcal{F}_n]$ is defined

as the unique r.v. s.t.

① $\mathbb{E}_n[X]$ is $\mathcal{F}_n$-m'ble

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \mathbb{E}_n[X](\omega) \mathbb{P}(\omega) = \sum_{\omega \in A} X(\omega) \mathbb{P}(\omega)$

prop (property of Condi Exp)

① $\mathbb{E}_n[X + \alpha Y] = \mathbb{E}_n[X] + \alpha \mathbb{E}_n[Y]$ (Linearity)

② (Tower Property) $n < m$. $\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]$

③ (Taking out what's known) X is $\mathcal{F}_n$-m'ble, $\mathbb{E}_n[XY] = X \mathbb{E}_n[Y]$

④ X is $\mathcal{F}_n$-ind, $\mathbb{E}_n[X] = \mathbb{E}[X]$

⑤ (Ind Lem). X is $\mathcal{F}_n$-m'ble, $Y \perp\!\!\!\perp \mathcal{F}_n$. Then

$\mathbb{E}_n[f(X, Y)] = g(X)$ where $g(x) = \mathbb{E}[f(x, Y)]$

⑥ (Jensen's Ineq) φ: Convex. $\varphi(\mathbb{E}_n[X]) \le \mathbb{E}_n[\varphi(X)]$.

Def (Martingale) M is a mart if

① M is adapted, M_n is $\mathcal{F}_n$-m'ble for all n.

② $\mathbb{E}_n[M_m] = M_n$, $n < m$

2. Application to Finance

Ω, p, q, r, S

Def (self-financing) X is self-fin if ~~Def X~~

$$X_{n+1} = \Delta_n \cdot S_{n+1} + (1+r)(X_n - \Delta_n S_n).$$

Def (Arbitrage) $\exists X$: self-fin s.t. $X_0 = 0$, $\mathbb{P}(X_N \geq 0) = 1$,

$\mathbb{P}(X_N > 0) > 0$.

Def (RN). $\tilde{\mathbb{P}}$ is RN if

① $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ ($\tilde{\mathbb{P}}(A) = 0 \iff \mathbb{P}(A) = 0$)

② Under $\tilde{\mathbb{P}}$, $D_n S_n$ is a martingale.

EX $S_{n+1} = \begin{cases} u S_n & (\omega_{n+1} = H) \\ d S_n & (\omega_{n+1} = T) \end{cases}$. $\tilde{p} = \dfrac{(1+r)-d}{u-d}$, $\tilde{q} = \dfrac{u-(1+r)}{u-d}$

$(d < 1+r < u)$

Thm $\exists$ RN $\tilde{\mathbb{P}}$. $V_n = \dfrac{1}{D_n} \tilde{\mathbb{E}}_n [D_N V_N]$

EX $S_{n+1} = \begin{cases} u S_n & (\omega_{n+1} = H) \\ d S_n & (\omega_{n+1} = T) \end{cases}$ $\Delta_n = \dfrac{V_{n+1}(\omega_{n+1}=H) - V_{n+1}(\omega_{n+1}=T)}{S_{n+1}(\omega_{n+1}=H) - S_{n+1}(\omega_{n+1}=T)}$

EX (HW 4 #6) $r=0$, $S_0 = 100$. $V_5 = (S_5 - 80)^+$

$V_0 = ?$. $S_{n+1} = \begin{cases} S_n + 10 & (\omega_{n+1} = H) \\ S_n - 10 & (\omega_{n+1} = T) \end{cases}$

RN $\tilde{\mathbb{E}}_n [S_{n+1}] = S_n \Rightarrow (S_n + 10) \cdot \tilde{p} + (S_n - 10) \cdot \tilde{q} = S_n$

$\Rightarrow \tilde{p} = \tilde{q} = 1/2$

pricing $V_n = f_n(S_n)$, $N=5$ $f_5(x) = (x-80)^+$

~~suppose $V_n = f_n(S_n)$, for~~ suppose $V_{n+1} = f_{n+1}(S_{n+1})$.

$$V_n = \tilde{\mathbb{E}}_n[V_{n+1}] = \tilde{\mathbb{E}}_n[f_{n+1}(S_{n+1})]$$
$$= \tilde{\mathbb{E}}_n[f_{n+1}(S_n + X_{n+1})]$$

$$\left(\text{where } X_{n+1} = \begin{cases} 10 & \omega_{n+1} = H \\ -10 & \omega_{n+1} = T \end{cases} \right)$$

$$= f_n(S_n) \quad \text{where } f_n(x) = \tilde{\mathbb{E}}[f_{n+1}(x + X_{n+1})]$$
$$= \frac{1}{2} \cdot f_{n+1}(x+10) + \frac{1}{2} f_{n+1}(x-10).$$

$$\Rightarrow f_n(x) = \frac{f_{n+1}(x+10) + f_{n+1}(x-10)}{2}$$

$$V_0 = f_0(100) = \frac{1}{2}(f_1(110) + f_1(90)) = \frac{1}{4}(f_2(120) + 2 \cdot f_2(100) + f_2(80))$$
$$= \cdots = \cancel{\ } f_5(\cdot) = \cdots.$$

II. Continuous Time case

$\Omega = C[0, \infty)$, $\mathbb{P}$

def (BM) W is a BM if

① W is continuous ② $W_t - W_s \sim N(0, t-s)$. ③ $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

④ $W_0 = 0$

def (Filtration): $\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t$, $s \le t$ where

$\mathcal{F}_t = \{$ events that are generated by W_s, $0 \le s \le t \}$

$\mathcal{F}_0 = \{ A \mid \mathbb{P}(A) = 0,\ \mathbb{P}(A) = 1 \}$.

def (Condi Exp) A r.v $X : \Omega \to \mathbb{R}$, $\mathbb{E}_t[X] = \mathbb{E}[X \mid \mathcal{F}_t]$ is

the unique r.v. s.t.

① $\mathbb{E}_t[X]$ is $\mathcal{F}_t$-m'ble

② $\forall A \in \mathcal{F}_t$, $\mathbb{E}[\mathbb{E}_t[X] \cdot \mathbb{1}_A] = \mathbb{E}[X \cdot \mathbb{1}_A]$

Rmk $\mathbb{E}[X \mid \mathcal{F}]$ $\mathbb{E}[X \mid Y] = \mathbb{E}[X \mid \sigma(Y)]$

prop (properties of Condi Exp) Same with the discrete time case.

Def (Martingale) same "

$M^2 - \langle M, M \rangle$: conti mart.

* Prop M : Continuous mart $\Rightarrow$

Thm D : continuous adapted. $M_t = \int_0^t D_s \, dW_s$.

① M is a mart.

② $\langle M, M \rangle_t = \int_0^t D_s^2 \, ds$.

Thm (Ito Isometry) $E\left[\left(\int_0^t D_s \, dW_s\right)^2\right] = E\left[\int_0^t D_s^2 \, ds\right]$

Ex $\langle W, W \rangle_t = t$

Prop X is a semimartingale. $X = X_0 + M + B$

$\langle X, X \rangle = \langle M, M \rangle$

Def X is an Ito process $X_t = X_0 + \int_0^t \sigma_s \, dW_s + \int_0^t b_s \, ds$.

Ex $\langle X, X \rangle_t = \int_0^t \sigma_s^2 \, ds$.

Thm (Ito formula). X is a semimartingale, $f = f(t,x) \in C^2$,

then $f(t,X_t)$ is also a semimartingale,

$$df(t,X_t) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d\langle X,X\rangle_t .$$

Thm (Lévy characterization)

X is B.M $\Longleftrightarrow$ $X_0 = 0$, X is a mart, $\langle X,X\rangle_t = t$

Ex $M_t = \int_0^t \text{sign}(W_s)\,dW_s$. $\rightsquigarrow$ M : B.M.

Thm (Deterministic Ito Integral).

$M_t = \int_0^t \sigma_s\,dW_s$, σ_s : deterministic fct of s.

$\Rightarrow$ M is Gaussian $N(0, \int_0^t \sigma_s^2\,ds)$.

Ito on $e^{\lambda M_t - \frac{\lambda^2}{2}\int_0^t \sigma_s^2\,ds}$ $\to$ mart

Ex $M_t = \int_0^t s\,dW_s$ $\rightsquigarrow$ $M_t \sim N(0, \int_0^t s^2\,ds)$

$\sim N(0, \frac{t^3}{3})$

Ex $M_t = \int_0^t W_s\,dW_s$

Problems

Ex (2019 Mid #1) $X_t = \int_0^t W_s^2 \, dW_s$

- $\mathbb{E}[X_t] = 0$
- $\mathbb{E}[X_t^2] = \mathbb{E}[\langle X, X \rangle_t] = \mathbb{E}\left[\int_0^t W_s^4 \, ds\right] = \int_0^t \mathbb{E}[W_s^4] \, ds = t^3$.

$W_s \sim N(0, s)$, $W_s \overset{d}{=} \sqrt{s} \cdot Z$, $Z \sim N(0,1)$

$\mathbb{E}[W_s^4] = \mathbb{E}[s^2 \cdot Z^4] = s^2 \cdot \mathbb{E}[Z^4] = 3s^2$.

Ex (2020 Mid #5). Find $\mathbb{E}\left[W_t \cdot \int_0^t s^2 \, dW_s\right]$

$dX_t = \_\_\_ \, dt + t^2 \, dW_t$

$df(t, W_t) = (\partial_t f + \frac{1}{2} \partial_x^2 f) \, dt + \partial_x f \, dW_t$ (Ito Formula)

$f(t, x) = t^2 \cdot x$

$d\, t^2 W_t = (2t W_t + \frac{1}{2} \cdot 0) \, dt + t^2 \, dW_t$

$$\Rightarrow t^2 W_t = \int_0^t 2s W_s \, ds + \underline{\int_0^t s^2 \, dW_s}$$

$$\Rightarrow \int_0^t s^2 \, dW_s = t^2 W_t - \int_0^t 2s W_s \, ds$$

$$\Rightarrow W_t \int_0^t s^2 \, dW_s = t^2 W_t^2 - W_t \int_0^t 2s W_s \, ds$$

$$\Rightarrow \mathbb{E}\left[W_t \int_0^t s^2 \, dW_s\right] = \mathbb{E}[t^2 W_t^2] - \mathbb{E}\left[W_t \int_0^t 2s W_s \, ds\right]$$

$$= t^3 - \mathbb{E}\left[\int_0^t 2s W_t W_s \, ds\right]$$

$$= t^3 - \int_0^t 2s \, \mathbb{E}[W_t W_s] \, ds$$

$$\boxed{※ \ \mathbb{E}[W_r \cdot W_s] = r \wedge s}$$

$$= t^3 - \int_0^t 2s^2 \, ds = \frac{t^3}{3}.$$

Ex (HW#4 Prob 3) $\mathbb{E}[M_t] = 0$

M is a mart $\;\Longleftarrow\;$ M has an ind. incre.

$\;\not\Longrightarrow\;$ $M_t - M_s \perp\!\!\!\perp \mathcal{F}_s$

pf ($\Leftarrow$) $\mathbb{E}_s[M_t] = \mathbb{E}_s[M_t - M_s] + \mathbb{E}_s[M_s]$

$= \mathbb{E}[M_t - M_s] + M_s = M_s$

($\not\Rightarrow$) Counterexample. $M_t = \int_0^t W_s \, dW_s$, $\langle M, M\rangle_t = \int_0^t W_s^2 \, ds$

WTS $M_t - M_s$ is NOT ind of $\mathcal{F}_s$.

$\mathbb{E}_s[(M_t - M_s)^2]$

$= \mathbb{E}_s[M_t^2 - 2M_t M_s + M_s^2] = \mathbb{E}_s[M_t^2 + M_s^2] - 2\underbrace{\mathbb{E}_s[M_t M_s]}_{= M_s^2}$

$= \mathbb{E}_s[M_t^2 - M_s^2]$

$= \mathbb{E}_s[\langle M, M\rangle_t - \langle M, M\rangle_s]$

why? $M^2 - \langle M, M\rangle$ $\quad \mathbb{E}_s[M_t^2 - \langle M, M\rangle_t] = M_s^2 - \langle M, M\rangle_s$

$$= \mathbb{E}_s\left[\int_s^t W_u^2\,du\right]$$

$$= \int_s^t \mathbb{E}_s[W_u^2]\,du$$

$$= \int_s^t \underline{\mathbb{E}_s[W_u^2 - u]} + u\,du$$

$$= \int_s^t W_s^2 - s + u\,du = \underline{(t-s)\cdot(W_s^2 - s) + \frac{t^2 - s^2}{2}}$$

Goal : $\underline{\mathbb{E}[(M_t - M_s)^2 W_s^2]} \neq \mathbb{E}[(M_t-M_s)^2]\,\mathbb{E}[W_s^2]$ ✓

$$\mathbb{E}[(M_t-M_s)^2 W_s^2] = \mathbb{E}\left[W_s^2\,\mathbb{E}_s[(M_t-M_s)^2]\right]$$

$$= \mathbb{E}\left[W_s^2\cdot\left((t-s)(W_s^2-s) + \frac{t^2-s^2}{2}\right)\right]$$

$$= \cdots$$

Ex (2021 Mid #5) $M_t = \int_0^t s W_s \, ds$. $\mathbb{E}[M_t^2 - \langle M, M\rangle_t] = ?$

- $\langle M, M\rangle_t = 0$

Method1

- $\mathbb{E}[M_t^2] = \mathbb{E}\left[\left(\int_0^t s W_s \, ds\right)\left(\int_0^t r W_r \, dr\right)\right]$

$$= \mathbb{E}\left[\int_0^t \int_0^t s r W_s W_r \, ds \, dr\right]$$

$$= \int_0^t \int_0^t s r \, \mathbb{E}[W_s W_r] \, ds \, dr = \cdots\cdots = \frac{2t^5}{15}.$$

$$= \int_0^t \int_0^t s r \cdot (s \wedge r) \, ds \, dr$$

Method2

$$d\cancel{X}_t = (\cdots) dt + \cdots dW_t$$

$$d t^2 W_t = 2t W_t \, dt + t^2 \, dW_t$$

$$\Rightarrow M_t = \int_0^t s W_s \, ds = \frac{1}{2}\left(t^2 W_t - \int_0^t s^2 \, dW_s\right)$$

$$= \frac{1}{2} \int_0^t (t^2 - s^2) \, dW_s \sim N\left(0, \frac{2t^5}{15}\right)$$

$$\boxed{\mathbb{E}[e^{\lambda M_t}] = e^{\frac{\lambda^2}{2} \cdot \frac{2t^2}{15}}}$$