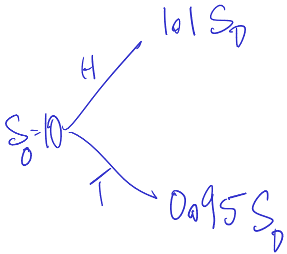
7. Review Problems

Problem 7.1 (From 2021 Midterm). Consider a discrete time market consisting of a bank and a stock. The bank pays interest rate $r = 5\%$ at every time period. Let S_n denote the stock price at time n, and we know $S_0 = \$10$. The stock price changes according to the flip of a fair coin: if the coin lands heads the stock price increases by 10% (i.e. $S_{n+1} = 1.1S_n$), and if the coin lands tails the stock price decreases by 5% (i.e. $S_{n+1} = 0.95S_n$). An option pays the holder $\boxed{S_N^3}$ at time $N = 5$. Find the arbitrage free price of this option at time $n = 1$. Also find the number of shares held in the replicating portfolio at time $n = 0$. Round your final answer two decimal places. (I recommend rounding intermediate steps to three decimal places.)

Bank ← $r = 5\%$

Stock ← $S_0 = \$10$



$u = 1.1$

$d = 0.95$

$r = 0.05$

Payoff $V_N = S_N^3$ $(N = 5)$

Find $V_1 = $ AFP at $n = 1$

Formula: V_n = AFP at time n $= \underbrace{\frac{1}{D_n} \tilde{E}_n (D_N V_N)}$

$$D_n = (1+r)^{-n}$$

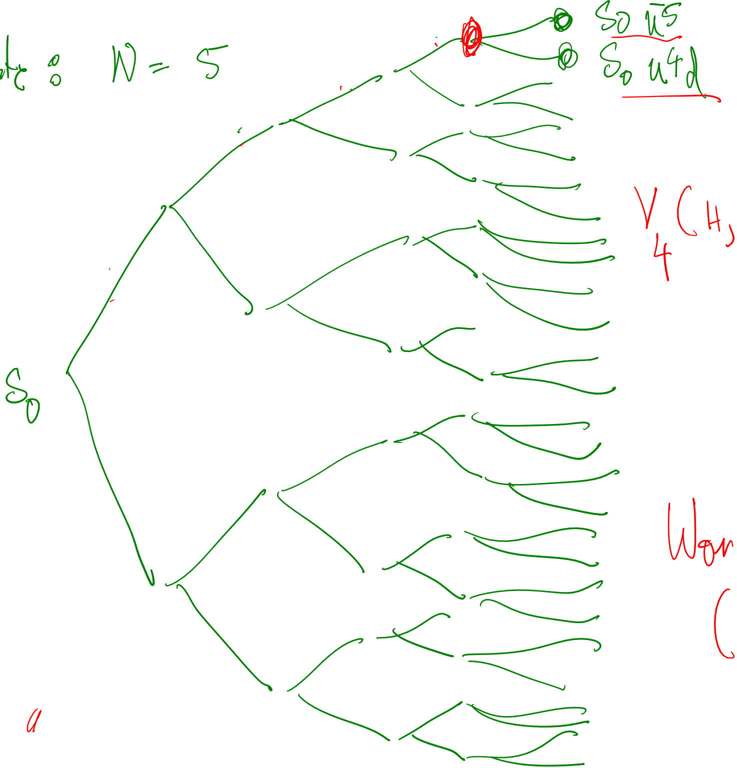
$\tilde{E} \rightarrow$ coin flips with prob. $\tilde{p}$ heads

$\tilde{q}$ tails

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1+r}{u-d}$$

n

Compute: $N = 5$



$$V_4(H,H,H,H,*) = \frac{1}{1+r}\tilde{E}_4(S_5)$$

$$= \frac{1}{1+r}\left(\tilde{p}\, S_0 u^5 + \tilde{q}\, S_0 u^4 d\right)$$

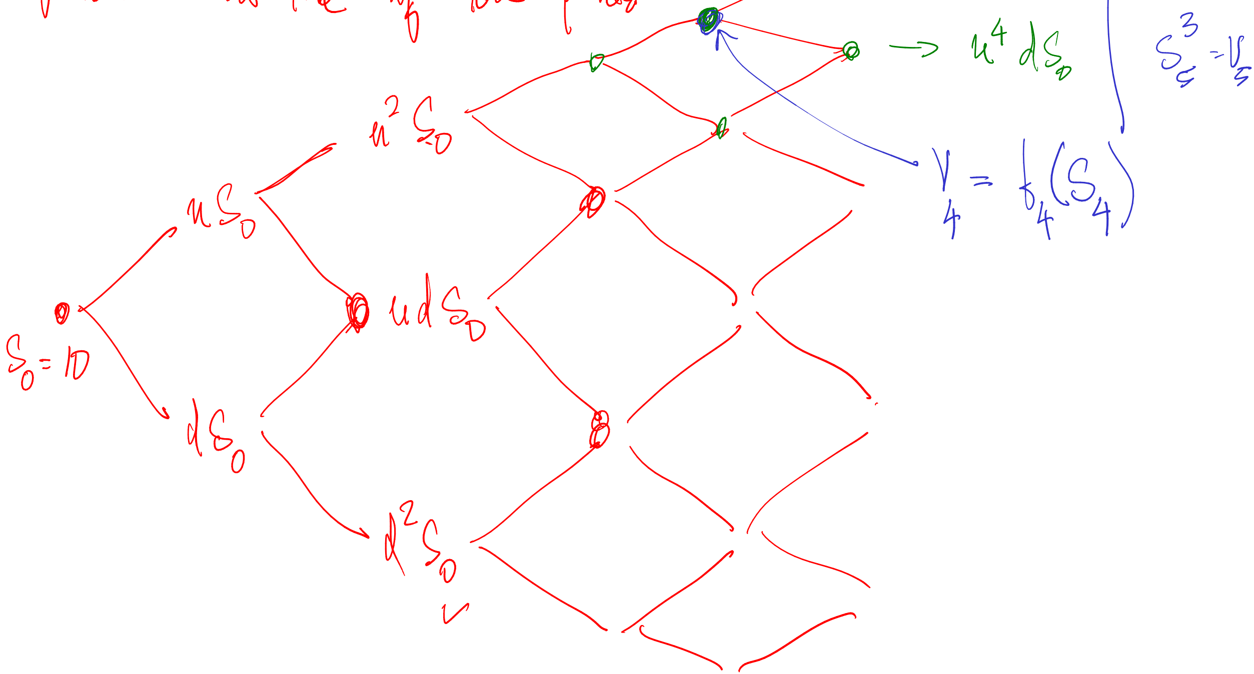
Works, but too much "work"
(Without a computer)

Better Strategy: From HW.

Know if $V_N = f_N(S_N)$

then $V_n = f_n(S_n)$ & $\boxed{f_n(x) = \dfrac{f_{n+1}(ux)\tilde{p} + f_{n+1}(xd)\tilde{q}}{1+r}}$

Use trick: Draw tree of stock prices



$$f_4(x) = \frac{f_5(ux)\tilde{p} + f_5(dx)\tilde{q}}{1+r} = \frac{(ux)^3\tilde{p} + (dx)^3\tilde{q}}{1+r}$$

$(x = S_4)$ & complete.

Even shorter method:

Know $\quad V_n = \frac{1}{D_n}\tilde{E}_n(D_N V_N)$

Used $D_n V_n$ is a mg under $\tilde{P}$

Know $D_n V_n = \tilde{E}_n\left(D_{n+1} V_{n+1}\right)$

$$D_n = (1+r)^{-n} \Rightarrow V_n = \frac{1}{1+r}\tilde{E}_n\left(\cancel{D_{n+1}}\, V_{n+1}\right)$$

$$V_{N-1} = \frac{1}{1+r}\tilde{E}_{N-1}\left(V_N\right) = \underbrace{\frac{1}{1+r}\tilde{E}_{N-1}\left(S_N^3\right)}$$

Let $X_n = \begin{cases} u & \text{if } n^{th} \text{ coin is heads} \\ d & \text{if } n^{th} \text{ coin is tails} \end{cases}$

Then $S_{n+1} = \underbrace{S_n}_{\mathcal{F}_n\text{-meas}} \underbrace{X_{n+1}}_{\text{indep of } \mathcal{F}_n.}$

So $\frac{1}{1+r}\tilde{E}_{N-1}\left(S_N^3\right) = \frac{1}{1+r}\tilde{E}_{N-1}\left(S_{N-1}^3 X_N^3\right)$

$$= \frac{1}{1+r} S_{N-1}^3 \, \tilde{E}_{\cancel{N-1}} X_N^3$$

(∵ S_{N-1} is $\mathcal{F}_{N-1}$ meas, X_N is indep)

$$\boxed{\Rightarrow \quad V_{N-1} = S_{N-1}^3 \left(\frac{u^3 \tilde{p} + d^3 \tilde{q}}{1+r} \right)}$$

Repeat: $V_{N-2} = \frac{1}{1+r}\tilde{E}_{N-2} V_{N-1} = \frac{u^3\tilde{p} + d^3\tilde{q}}{(1+r)^2} \tilde{E}_{N-2} S_{N-1}^3$

$$= \frac{(u^3\tilde{p} + d^3\tilde{q})}{(1+r)^2} \, S^3_{N-2} \, (u^3\tilde{p} + d^3\tilde{q})$$

$$= \left(\frac{u^3\tilde{p} + d^3\tilde{q}}{1+r}\right)^2 S^3_{N-2}$$

$$\therefore V_1 = V_{N-\underline{4}} = \left(\frac{u^3\tilde{p} + d^3\tilde{q}}{1+r}\right)^4 S^3_1$$

Problem 7.2. If $0 \leqslant r \leqslant \underline{s \leqslant t}$, find $\boldsymbol{E}(W_s W_t)$ and $\boxed{\boldsymbol{E}(W_r W_s W_t)}$.

Compute $E(W_s W_t) = s \wedge t = s$

Check: $E(W_s W_t) = E\Big(\underline{W_s}\underbrace{(W_t - W_s)}_{\text{indep}} + W_s^2\Big)$

$= \underline{EW_s}\, E(W_t - W_s) + EW_s^2$

$= 0 + s \quad (W_s \sim N(0,s))$

Check 2: $E(W_s W_t) = E\, E_s(W_s W_t)$ (tower)

$$= E\Big(W_s \underbrace{E_s W_t}_{mg}\Big) = E(W_s \cdot W_s) = s.$$

Compute $E(W_r W_s W_t)$

$$= E\big(E_r\, W_r W_s W_t\big)$$

$$= E\Big(W_r\, E_r\big(W_s W_t\big)\Big)$$

$$= E\Big(W_r\, E_r\, E_s\big(W_s W_t\big)\Big)$$

$$= E\Big(W_r\, E_r\big(W_s\, E_s W_t\big)\Big)$$

$$= E\left(W_r \, E_r\left(W_s^2 \right) \right)$$

$$= E\left(W_r \, E_r\left(\underbrace{W_s^2 - s}_{\text{mg}} + s \right) \right)$$

$$= E\left(W_r \left(W_r^2 - r + s \right) \right)$$

$$= E\, W_r^3 + E\, W_r (s - r) = 0.$$

Problem 7.3. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2}\, ds\,, \quad \boxed{Y_t = \exp\Big(\int_0^t W_s\, ds\Big)\,,} \quad Z_t = tX_t^2$$

Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

Ito decompose X:

Write $\quad X_t = f(t, W_t) \quad$ where $\boxed{f(t,x) = \int_0^x e^{-s^2}\, ds}$

Need $\partial_t f$, $\partial_x f$, $\partial_x^2 f$ to exist

$\partial_t f = 0$

$\partial_x f = e^{-x^2}$ $\quad \left(\text{FTC: } \partial_x \int_0^x g(s)\, ds = g(x)\right)$

$$\partial_x^2 f = e^{-x^2}(-2x).$$

Itô: $dX = d f(t, W_t) = \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2}\partial_x^2 f \, d[W, W]$

$$= 0 + e^{-W_t^2} dW_t + \frac{1}{2}\left(e^{-W_t^2}(-2W_t)\right) dt$$

$$dX_t = \underline{e^{-W_t^2}} dW_t - W_t e^{-W_t^2} dt$$

$$\Rightarrow X_t = X_0 + \int_0^t e^{-W_s^2} dW_s - \int_0^t W_s e^{-W_s^2} ds \quad \textit{4}$$

Mg Finite 1^{st} var.

Remember $\longrightarrow$ Ito ints are Mg's

& Riemann int are finite 1^{st} var.

$$Y_t = \underbrace{\exp\left(\int_0^t W_s \, ds\right)}$$ Find Ito decomp.

$$Y_t = f(t, W_t) \quad \& \quad f(t,x) = \begin{cases} \exp\left(\int_0^t W_s \, ds\right) \\ \cancel{\exp\left(\int_0^t x \, ds\right)} \end{cases}$$

$$\partial_x f(t,x) = 0$$

$$\partial_x^2 f(t,x) = 0$$

$$\partial_t f(t,x) = \exp\left(\int_0^t W_s\,ds\right) \frac{d}{dt}\int_0^t W_s\,ds = W_t$$

$$= f(t,x) \quad W_t.$$

$$\therefore dY = \partial_t f\,dt + \underbrace{\partial_x f}_{0}\,dW_t + \frac{1}{2}\underbrace{\partial_x^2 f}_{0}\,d[W,W]$$

✓

$$= f(t, W_t) \cdot W_t \, dt = Y_t W_t \, dt$$

$$\Rightarrow Y_t = Y_0 + \underbrace{\int_0^t Y_s W_s \, ds}_{\text{Finite 1st var}} + \underbrace{0}_{\text{Mg.}}$$

Problem 7.4. Define the processes X, Y by

$$X_t \stackrel{\text{def}}{=} \int_0^t W_s \, ds \,, \qquad \boxed{Y_t \stackrel{\text{def}}{=} \int_0^t W_s \, dW_s \,.}$$

Given $0 \leqslant s < t$, compute $\boldsymbol{E} X_t$, $\boldsymbol{E} Y_t$, $\boldsymbol{E}_s X_t$, $\boldsymbol{E}_s Y_t$.

$$E X_t = E \int_0^t W_s \, ds = \int_0^t (E W_s) \, ds = 0$$

$$E_s X_t = E_s \int_0^t W_r \, dr = \int_0^t E_s W_r \, dr$$

$$= \int_0^s E_s W_r \, dr + \int_s^t E_s W_r \, dr$$

Rules:

$$\to E \int_0^t b_s \, ds = \int_0^t E_s b_s \, ds$$

$$E_s \int_0^t b_r \, dr = \int_0^t E_s b_r \, dr$$

$$E_s \int_0^t \sigma_r \, dW_r = \int_0^s \sigma_r \, dW_r$$

Ito int are mg's.

$$= \int_0^s W_r\,dr + \int_s^t W_s\,dr$$

$$= \int_0^s W_r\,dr + W_s(t-s)$$

$$E\int_0^t W_r\,dW_r = E\int_0^0 W_r\,dW_r = 0 \qquad (Mg)$$

$$E_s\int_0^t W_r\,dW_r = \int_0^s W_r\,dW_r \qquad (Mg)$$

Problem 7.5. Let $M_t = \int_0^t W_s \, dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\Big(M_t - \int_0^t f(s, W_s) \, ds\Big)$$

is a martingale.

Problem 7.6. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 \, dt$.

$$X_t = \int_0^t \sigma_u \, dW_u \, .$$

(1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leqslant s < t$ compute $\boldsymbol{E} e^{\lambda M_t}$ and $\boldsymbol{E}_s e^{\lambda M_t - M_s}$
(2) If $r \leqslant s$ compute $\boldsymbol{E} \exp(\lambda M_r + \mu(M_t - M_s))$.
(3) What is the joint distribution of $(M_r, M_t - M_s)$?
(4) *(Lévy's criterion)* If $d[M, M]_t = dt$, then show that M is a standard Brownian motion.

Problem 7.7. Define the process X, Y by

$$X = \int_0^t s\, dW_s\,, \quad Y = \int_0^t W_s\, ds\,.$$

Find a formula for $\boldsymbol{E}X_t^n$ and $\boldsymbol{E}Y_t^n$ for any $n \in \mathbb{N}$.

$\boxed{\text{Claim: Both } X_t \text{ \& } Y_t \text{ are normal!}}$

Can find EX_t^n & EY_t^n if I know EX_t & EX_t^2 & EY_t & EY_t^2.

Find EX_t & EX_t^2:

a

$$E X_t = E \int_0^t s \, dW_s = 0$$

$$E X_t^2 = E \left(\int_0^t \underline{\underline{s}} \, d\underline{\underline{W_s}} \right)^2 = E \int_0^t s^2 \, ds = \frac{t^3}{3}$$

for Y:

$$E Y_t = E \left(\int_0^t W_s \, ds \right) = \int_0^t E W_s \, ds = 0$$

$$E Y_t^2 = E \left(\int_0^t W_s \, ds \right)^2$$

$$\Rightarrow E\left[\left(\int_0^t W_s \, ds\right)\left(\int_0^t W_s \, ds\right)\right]$$

$$= E\left[\left(\int_0^t W_s \, ds\right)\left(\int_0^t W_r \, dr\right)\right]$$

$$= E \int_{s=0}^{t} \int_{r=0}^{t} W_s W_r \, ds \, dr = \int_{s=0}^{t} \int_{r=0}^{t} E(W_s W_r) \, ds \, dr$$

4

$$= \int_{s=0}^{t} \int_{r=0}^{t} (s \wedge r)\, ds\, dr$$ & compute.

Why are X & Y normal:

Not random

Normal

$$X_t = \int_0^t s\, dW_s = \lim \sum \underbrace{t_k \overbrace{(W_{t_{k+1}} - W_{t_k})}^{\text{Normal}}}_{\text{sum of ind normals.}} \longrightarrow \text{Normal.}$$

2

$$Y_t = \int_0^t W_s \, ds = \underbrace{\lim_{\|P\| \to 0} \underbrace{\sum \underbrace{W_{t_k}}_{\text{normal}} (t_{k+1} - t_k)}_{\text{sum of normals}}}_{\text{Normal.}}$$

u

$$\mathbb{E}_s \int_0^t \sigma_r \, dW_r = \int_0^t \mathbb{E}_s \left(\sigma_r \, dW_r \right)$$

may not make sense!

u

Problem 7.8. Let $M_t = \int_0^t W_s \, dW_s$. For $s < t$, is $M_t - M_s$ independent of $\mathcal{F}_s$? Justify.

Problem 7.9. Determine whether the following identities are true or false, and justify your answer.

(1) $e^{2t}\sin(2W_t) = 2\int_0^t e^{2s}\cos(2W_s)\,dW_s$.

(2) $|W_t| = \int_0^t \operatorname{sign}(W_s)\,dW_s$. (Recall $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 0$ if $x = 0$.)