1. $I^2 - A$ mart $\qquad \mathbb{E}_s[I_t^2 - A_t] = I_s^2 - A_s$ ✪

① $s \le t \le t_1$ ✓

② $s \le t_1 \le t$ ✓

③ $t_1 \le s \le t$ ✓

① + ③ $\Longrightarrow$ ②

$\because$ $s \le t_1 \le t$

✪ : $\mathbb{E}_s[I_t^2 - A_t] = \mathbb{E}_s[\mathbb{E}_{t_1}[I_t^2 - A_t]]$

$$\overset{③}{=} \mathbb{E}_s[I_{t_1}^2 - A_{t_1}]$$

$$\overset{①}{=} I_s^2 - A_s$$

③ $t_1 \le s \le t$

$$I_t^2 - I_s^2 = \underbrace{\varphi_1(W_t - W_s)}_{I_t - I_s} \underbrace{\left(2\varphi_0 W_{t_1} + \varphi_1(W_t - W_{t_1}) + \varphi_1(W_s - W_{t_1})\right)}_{I_t + I_s}$$

$$\mathbb{E}_s[I_t^2 - I_s^2] = 2\varphi_1 \cdot \varphi_0 \cdot W_{t_1} \cdot \cancel{\mathbb{E}[W_t - W_s]} \to 0$$

$$+ \varphi_1^2 \, \mathbb{E}_s[(W_t - W_s) \cdot (W_t - W_{t_1})] \quad (*)$$

$$= \varphi_1^2 \cdot (W_s - W_{t_1}) \cdot \cancel{\mathbb{E}[W_t - W_s]} \to 0$$

$$\circledast = \mathbb{E}_s[(W_t - W_s)\cdot(W_t - W_s)] = t-s$$

$$+ \mathbb{E}_s[(W_t - W_s)\cdot\underline{(W_s - W_{t_1})}] + 0$$

$$\mathbb{E}_s[I_t^2 - I_s^2] = \varphi_1^2 \cdot (t-s) = A_t - A_s \quad \checkmark$$

3.(b) $S_t = S_0 \cdot e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t}$

$$= h(t, W_t)$$

$$h(t,x) := S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma x}$$

$$dS_t = \partial_t h \cdot dt + \partial_x h \cdot dW_t + \frac{1}{2}\partial_x^2 h \cdot d\underline{\langle W, W\rangle_t}$$

$$= (\partial_t h + \frac{1}{2}\partial_x^2 h)\,dt + \partial_x h\, dW_t.$$

... computation

5.(b) $X_t = \int_0^t \text{sign}(W_s)\, dW_s$

$\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t^2] = \mathbb{E}\left[\int_0^t \text{sign}(W_s)^2\, ds\right]$

$= t$

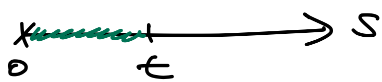
suppose "$X_t = |W_t| \geq 0$"

$\mathbb{E}[X_t] = 0 \Rightarrow X_t = 0 \Rightarrow \mathbb{E}[X_t^2] = 0$

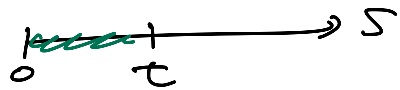
$M_t = \int_0^t H_s\, dW_s \Leftarrow \mathbb{E}\left[\int_0^t H_s^2\, ds\right] < \infty$

is a mart

$\int_0^t f(s)\, ds$



$\int_0^t H_s\, dW_s$



3.(f) $s<t$ $\mathbb{E}_s[f(W_t)] = g(W_s)$.

"we have to find a function g"

$$\mathbb{E}_s[f(W_t)] = \mathbb{E}_s[f(\underbrace{W_t - W_s}_{\perp\!\!\!\perp \mathcal{F}_s} + \underbrace{W_s}_{\mathcal{F}_s\text{-m'ble}})]$$

Ind Lem

$$= g(W_s) \quad \text{where}$$

$$g(y) = \mathbb{E}[f(W_t - W_s + y)]$$

$X \sim N(0,1)$

$$\Sigma = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases}$$

$\mathbb{E}[X] = 0$

$\mathbb{E}[X^2] = 1$

$\mathbb{E}[\Sigma] = 0$

$\mathbb{E}[\Sigma^2] = 1$

But X, Σ different distribution

3(a) $S_t = S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t}$

$s \le t$ $\frac{S_t}{S_s} = e^{(\alpha - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)}$

$\mathbb{E}_s[f(S_t)]$

$= \mathbb{E}_s[f(\underbrace{S_s}_{\mathcal{F}_s\text{-m'ble}} \cdot \underbrace{e^{(\alpha - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)}}_{\mathcal{F}_s\text{-Ind}})]$

Ind Lem

X : $\mathcal{F}$-m'ble

Y : Ind of $\mathcal{F}$.

$\mathbb{E}[f(X, Y) | \mathcal{F}] = g(X)$ where

$g(x) = \mathbb{E}[f(x, Y)]$

$= g(S_s)$ where

$g(x) = \mathbb{E}[f(x e^{(\alpha - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s)})]$

pdf of $N(0, t-s)$: $\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}$

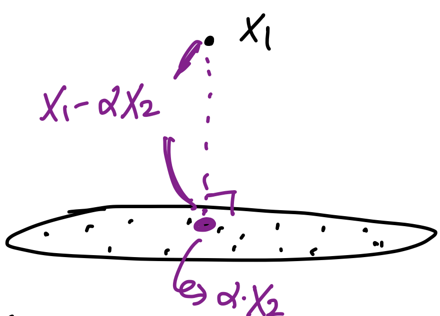
$$= \int_{-\infty}^{\infty} f(\cdots)\ \text{pdf}\ dy$$

4.(a) $W_t^4 = h(W_t)$ where $h(x) = x^4$.

$$dW_t^4 = h'(W_t) \cdot dW_t + \frac{1}{2} h''(W_t)\, \underbrace{d\langle W, W\rangle_t}_{=t}$$

$h'(x) = 4 \cdot x^3 \qquad h''(x) = 12 \cdot x^2$

2(a) $E[X_1 | X_2]$



(X_1, X_2) : Gaussian vector.

$\Rightarrow (Y, X_2)$: Gaussian vector.