Recall: Semi mg: $X_t = X_0 + \underbrace{B_t}_{\text{Finite 1st var}} + \underbrace{M_t}_{\text{mg}}$ (cts adapted)

$$B_t = \underbrace{\int_0^t b_s \, ds}_{\text{R int}} , \quad M_t = \underbrace{\int_0^t \sigma_s \, dW_s}_{\text{Ito int.}}$$

Notation: $dX_t = b_t \, dt + \sigma_t \, dW_t$

Notation: $\int_0^t D_s \, dX_s = \int_0^t D_s b_s \, ds + \int_0^t D_s \sigma_s \, dW_s.$

Theorem (Itô's formula, Theorem 6.29). *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \boxed{\frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t}$$

Remark 6.37. If $dX_t = b_t\, dt + \sigma_t\, dW_t$ then

$$df(t, X_t) = \Big(\partial_t f(t, X_t) + b_t \partial_x f(t, X_t) + \frac{1}{2}\sigma_t^2 \partial_x^2 f(t, X_t)\Big)\, dt + \partial_x f(t, X_t)\sigma_t\, dW_t\,.$$

$f \in C^{1,2}$ means: $f = f(t, x)$

$\to$ $\partial_t f$ exists & is cts

$\partial_x f$ & $\partial_x^2 f$ exist & are cts

$dX_t = b_t\, dt + \sigma_t\, dW$

$\Rightarrow d[X,X]_t = \sigma_t^2\, dt$

Intuition behind Itô's formula.

Itô $\iff$ $f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\, dt + \int_0^T \partial_x f(t, X_t)\, dX_t$

$+ \frac{1}{2} \int_0^T \partial_x^2 f(t, X_t)\, d[X, X]_t.$

Consider a simple case: $f(t, x) = f(x)$

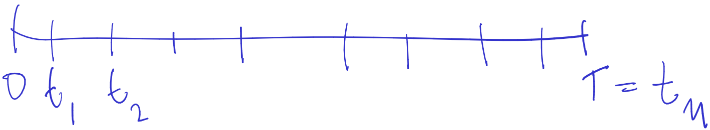
$X_t = W_t.$

Itô $\iff$ $f(W_T) - f(W_0) = \int_0^T \partial_x f(W_t)\, dW_t + \frac{1}{2} \int_0^T \partial_x^2 f(W_t)\, dt.$

Intuition behind Itô's formula.

Notation: Write $\partial_x f = f'$

Taylor's theorem: $f(x+h) - f(x) = h\, f'(x) + \frac{h^2}{2} f''(x) \boxed{+ O(h^3)}$



$$f(W_T) - f(W_0) = \sum_{0}^{n-1} f(W_{t_{k+1}}) - f(W_{t_k})$$

Taylor $\displaystyle\sum_{k=0}^{n-1} f'(W_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2} f''(W_{t_k})(W_{t_{k+1}} - W_{t_k})^2$

+ small.

$$\boxed{\Delta_k W = W_{t_{k+1}} - W_{t_k}.} = \underbrace{\sum_{k=0}^{n-1} f'(W_{t_k})\,\Delta_k W}_{\downarrow \|P\| \to 0} + \underbrace{\frac{1}{2}\sum_{k=0}^{n-1} f''(W_{t_k})\,(\Delta_k W)^2}_{\approx \frac{1}{2}\sum_{k=0}^{n-1} f''(W_{t_k})(t_{k+1} - t_k)} + \text{small}$$

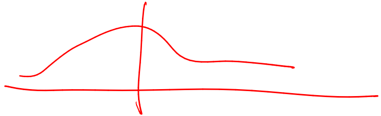
$$\int_0^T f'(W_t)\,dW_t$$

$$\underbrace{\frac{1}{2}\sum_{k=0}^{n-1} f''(W_{t_k})(t_{k+1} - t_k)}_{\downarrow \|P\| \to 0}$$

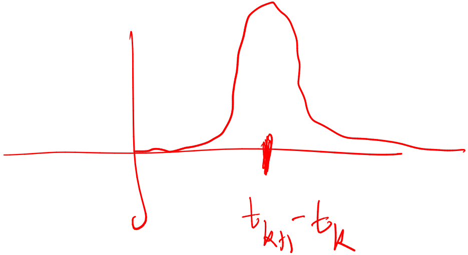
$$\frac{1}{2}\int_0^T f''(W_t)\,dt$$

$\Rightarrow$ Ito !!

Know $\Delta_k W \sim N(0, t_{k+1} - t_k)$



$$(\Delta_k W)^2 \sim N(0, \underbrace{t_{k+1}}_{} - t_k)^2$$



$$\mathrm{Var}(\Delta_k W) = 2\underbrace{(t_{k+1} - t_k)^2}_{\longrightarrow 0}$$

Example 6.38. Find the quadratic variation of W_t^2.

Last time

Example 6.39. Find $\boxed{\int_0^t W_s \, dW_s.}$

Method: Guess a fn $f = f(t,x)$

so that $d\,f(t, W_t) = \underline{\quad}\, dt + \boxed{W_t \, dW_t}$

& integrate.

Note $d\,f(t, W_t) = \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2} \partial_x^2 f \, dt$

Choose $f(t,x) = \dfrac{x^2}{2}.$

$$\partial_t f = 0$$
$$\partial_x f = x$$
$$\partial_x^2 f = 1$$

$$d\left(\frac{W_t^2}{2}\right) = d f(t, W_t)$$
$$= \partial_t f \, dt + \partial_x f \, dW_t + \frac{1}{2} \partial_x^2 f \, d[W, W]_t$$
$$= 0 + W_t \, dW_t + \frac{1}{2} dt$$

$$\Rightarrow \frac{W_T^2}{2} - 0 = \int_0^T W_t \, dW_t + \frac{T}{2}.$$

$$\boxed{\Rightarrow \int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}}$$

Example 6.40. Let $M_t = W_t$, and $N_t = W_t^2 - t$.

- We know M, N are martingales.
- Is MN a martingale?

Opt ① $\longrightarrow$ $M_t N_t = W_t^3 - t W_t$ & Compute $E_s(\quad)$

$$E_s(W_t^3) = E_s((W_t - W_s + W_s)^3) \text{ \& expand \& suffer.}$$

Better way: Use Ito:

Compute $d(M_t N_t) = d(W_t^3 - t W_t)$

Choose $f(t, x) = x^3 - tx$

$$\partial_t f = -x$$
$$\partial_x f = 3x^2 - t$$
$$\partial_x^2 f = 6x$$

Ito: $$d(W_t^3 - tW_t) = \partial_t f\, dt + \partial_x f\, dW + \frac{1}{2}\partial_x^2 f\, d[W,W]$$
$$= -W_t\, dt + (3W_t^2 - t)\, dW + \frac{1}{2} 6 W_t\, dt$$
$$= \underbrace{(3W_t - W_t)}\, dt + (3W_t^2 - t)\, dW_t$$

Coeff of $dt \neq 0 \Rightarrow$ MN can NOT be a mg.

Example 6.41. Let $X_t = t\sin(W_t)$. Is $X_t^2 - [X,X]_t$ a martingale?

Example 6.42. Say $dM_t = \sigma_t dW_t$. Show that $M^2 - [M, M]$ is a martingale.

($M \to$ mg) Check: $X_t = M_t^2 - [M,M]_t$,

$f(t,x) = x^2 - [M,M]_t$

(Recall: $[M,M]_t = \int_0^t \sigma_s^2 \, ds$

$\Leftrightarrow \boxed{d[M,M] = \sigma_t^2 \, dt}$ (diff))

Apply Ito: $d f(t,M) = \partial_t f dt + \partial_x f \, dM + \boxed{\frac{1}{2} \partial_x^2 f \, d[M,M]}$.

$$= \cancel{-\sigma_t^2\, dt} + 2M_t\, dM_t + \frac{1}{2}\cdot 2\, \cancel{\sigma_t^2\, dt}$$

$$\partial_t f = -\sigma_t^2$$

$$\boxed{\partial_x f = 2x}$$

$$\partial_x^2 f = 2.$$

$$= 2M_t\, \sigma_t\, dW_t$$

No dt term $\rightarrow$ is a mg!

Theorem 6.43 (Lévy's criterion)**.** *Let M be a continuous martingale such that* $\boxed{M_0 = 0}$ *and* $\boxed{[M, M]_t = t.}$ *Then M is a Brownian motion.*

Knows W → ① is cts
② is a mg
③ $d[W,W]_t = dt$

I.e. $M_t - M_s \sim N(0, t-s)$

& $M_t - M_s$ is ind of $\mathcal{F}_s$.

① Show $M_t \sim N(0,t)$

Will show $MGF(M_t) = MGF(N(0,t))$

$\varphi_\lambda(t) = E\, e^{\lambda M_t}$

X is a RV.

Then $\varphi = MGF$ of X

$\varphi(\cancel{t}\lambda) = E\, e^{\cancel{t}\lambda X}$

$\varphi_\lambda(N(0,t)) = e^{\lambda^2 t/2}$

Apply Itô to $d\left(e^{\lambda M_t}\right)$:

Choose $f(t,x) = e^{\lambda x}$

$\partial_t f = 0$

$\partial_x f = \lambda e^{\lambda x}$

$\partial_x^2 f = \lambda^2 e^{\lambda x}$

$$\Rightarrow d\left(e^{\lambda M_t}\right) = \underline{\partial_t f}\, dt + \partial_x f\, dM +$$

$$+ \frac{1}{2}\partial_x^2 f\, d[M,M]$$

$$= 0 + \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} e^{\lambda M_t} dt$$

(Given $d[M,M] = dt$)

$$\Rightarrow \underline{\underline{E e^{\lambda M_T}}} - 1 = E\int_0^T \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} E\int_0^T e^{\lambda M_t} dt$$

Say $dM = \sigma\, dW_t$

$$\Rightarrow E\int_0^T \lambda e^{\lambda M_t}\, dM_t = E\int_0^T \lambda e^{\lambda M_t} \sigma_t\, dW_t$$

$$= 0 \quad (\because \text{Itô int is a mg}).$$

$$\varphi_\lambda(t) = E e^{\lambda M_t}$$

$$\Rightarrow \varphi_\lambda(T) - 1 = 0 + \frac{\lambda^2}{2} E\int_0^T e^{\lambda M_t}\, dt$$

$$= \frac{\lambda^2}{2} \int_0^T \underbrace{E e^{\lambda M_t}}_{\varphi_\lambda(t)} dt$$

$$\varphi_\lambda(T) - 1 = \frac{\lambda^2}{2} \int_0^T \varphi_\lambda(t)\, dt$$

diff wrt T : $\varphi_\lambda'(T) = \frac{\lambda^2}{2} \varphi_\lambda(T)$

$$\Rightarrow \varphi_\lambda(T) = \varphi_\lambda(0) \cdot \exp\left(\frac{\lambda^2}{2} T\right)$$

$$\Rightarrow \varphi_\lambda(T) = e^{\frac{\lambda^2}{2}T} = \text{MGF of normal!!}$$

$$\Rightarrow M_t \sim N(0, t)$$

Try same strategy & show $M_t - M_s \sim N(0, t-s)$

& $M_t - M_s$ is ind of $\mathcal{F}_s$.

Let $\varphi(t) = E_s\, e^{\lambda(M_t - M_s)}$

Compute φ: (Ito $\Rightarrow$)

$$d\left(e^{\lambda M_t}\right) = \lambda e^{\lambda M_t} dM_t + \frac{\lambda^2}{2} e^{\lambda M_t} dt$$

Int from s to t:

$$e^{\lambda M_t} - \underline{\underline{e^{\lambda M_s}}} = \lambda \int_s^t e^{\lambda M_r} dM_r + \frac{\lambda^2}{2} \int_s^t e^{\lambda M_r} dr$$

$$\Rightarrow E_s e^{\lambda(M_t - M_s)} - 1 = \lambda E_s \int_s^t e^{\lambda(M_r - M_s)} \underline{\underline{dM_r}} + \frac{\lambda^2}{2} E_s \int_s^t e^{\lambda(M_r - M_s)} dr$$

adapted so OK.

$$\boxed{\varphi(t) = E_s\, e^{\lambda(M_t - M_s)}}$$

$$\Rightarrow \varphi(t) - 1 = \lambda \underbrace{\int_s^s (\quad)}_{0} + \frac{\lambda^2}{2}\int_s^t E_s\, e^{\lambda(M_r - M_s)}\, dr$$

$$\boxed{\Rightarrow \varphi(t) = 1 + \frac{\lambda^2}{2}\int_s^t \varphi(r)\, dr}$$

$\leftarrow$ same eq as before

$$\Rightarrow \varphi(t) = e^{\frac{\lambda^2}{2}(t-s)} = \text{MGF of normal.}$$

$$\Rightarrow \quad E_s\, e^{\lambda(M_t - M_s)} = e^{\frac{\lambda^2}{2}(t-s)}$$

$$\Rightarrow \quad E\, e^{\lambda(M_t - M_s)} = E\, E_s\, e^{\lambda(M_t - M_s)} = \underbrace{e^{\frac{\lambda^2}{2}(t-s)}}_{\text{MGF of } N(0,\,t-s).}$$

Indep:

X & Y are indep $\iff E\, e^{\lambda X + \mu Y} = E\, e^{\lambda X}\, E\, e^{\mu Y}$

If X is $\mathcal{F}_s$ meas

$$E\left(e^{mX + \lambda(M_t - M_s)}\right)$$

$$= E\, E_s\left(e^{\mu X}\, e^{\lambda(M_t - M_s)}\right)$$

$$= \underbrace{E\, e^{\mu X}}_{\text{MGF}(X)}\, E_s\, e^{\lambda(M_t - M_s)} \to e^{\frac{\lambda^2}{2}(t-s)}$$

$\Rightarrow$ Indep!.