Def Quadratic variation.

X: continuous, adapted

$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_p = T\}$, $\|\mathcal{P}\| = \max_{1 \le i \le p} |t_i - t_{i-1}|$

$$[X, X]_T = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{p} (X_{t_i} - X_{t_{i-1}})^2$$

EX $[W, W]_t = t$

Prop M: continuous mart, "$\mathbb{E}[M_t^2] < \infty$"

$\Rightarrow$ $M_t^2 - [M, M]_t$: continuous mart.

Rmk Unique. If A: continuous, adapted, increasing, $A_0 = 0$

s.t $M_t^2 - A_t$ is a mart, then $A_t = [M, M]_t$

EX $W_t^2 - t$ is also a mart

Q what is Q.V. of $W_t^2 - t$? $X_t = W_t^2 - t$

To compute $\langle X, X \rangle_t$, find A s.t $X_t^2 - A_t$ is a mart.

Def Ito Integral

D: continuous, adapted. $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_p = T\}$

$$\int_0^T D_s \, dW_s = \lim_{\|\mathcal{S}\| \to 0} \sum_{i=0}^{p} D_{t_{i-1}} (W_{t_i} - W_{t_{i-1}})$$

Thm If " $\mathbb{E}\left[\int_0^T D_s^2 \, ds\right] < \infty$ ", then

(1) $M_t = \int_0^t D_s \, dW_s$, $(M_t)_{0 \le t \le T}$ is a mart.

i.e. $\mathbb{E}_s[M_t] = M_s$ for all $0 \le s \le t \le T$.

(2) $\langle M, M \rangle_t = \int_0^t D_s^2 \, ds$

Cor (Ito Isometry) $\mathbb{E}\left[\left(\int_0^t D_s \, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t D_s^2 \, ds\right]$

pf $M_t := \int_0^t D_s \, dW_s$, $\langle M, M \rangle_t = \int_0^t D_s^2 \, ds$

we know that $M_t^2 - \langle M, M \rangle_t$ is a mart.

$$\mathbb{E}[M_t^2 - \langle M, M \rangle_t] = \mathbb{E}[M_0^2 - \langle M, M \rangle_0] = 0 \Rightarrow \mathbb{E}[M_t^2] = \mathbb{E}[\langle M, M \rangle_t]$$

<u>EX</u> $M_t = \int_0^t s\, dW_s$.

- $\langle M, M\rangle_t = \int_0^t s^2\, ds = \frac{t^3}{3}$.
- $\mathbb{E}[M_t] = \mathbb{E}[M_0] = 0$
- $\mathrm{var}(M_t) = \mathbb{E}[(M_t - \mathbb{E}[M_t])^2] = \mathbb{E}[M_t^2]$

$$= \mathbb{E}\left[\left(\int_0^t s\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t s^2\, ds\right] = \mathbb{E}\left[\frac{t^3}{3}\right] = \frac{t^3}{3}$$

<u>EX</u> $M_t = \int_0^t W_s\, dW_s$.

- $\langle M, M\rangle_t = \int_0^t W_s^2\, ds$
- $\mathbb{E}[M_t] = \mathbb{E}[M_0] = 0$
- $\mathrm{var}(M_t) = \mathbb{E}[M_t^2] = \mathbb{E}\left[\left(\int_0^t W_s\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t W_s^2\, ds\right]$

$$= \int_0^t \mathbb{E}[W_s^2]\, ds \quad \text{(Fubini)}$$

$$= \int_0^t s\, ds = \frac{t^2}{2}.$$

$$\int_0^1 \int_0^2 x \cdot \sin(y)\, dx\, dy$$

$$= \int_0^2 \int_0^1 x \cdot \sin(y)\, dy\, dx$$

<u>Def</u> Semimartingale

$X = X_0 + M + B$, M: martingale, B: finite 1^{st} order variation

<u>Prop</u> $X = X_0 + M + B$. Then $\langle X, X\rangle_t = \langle M, M\rangle_t$

<u>Rmk</u> X: semimartingale. $X_t^2 - \langle X, X\rangle_t$ may not be a mart.

<u>Ex</u> $X_t = \underline{t}^2 + \underline{\int_0^t W_s^2 \, dW_s}$

$\langle X, X\rangle_t = \int_0^t W_s^4 \, ds$. Q. $X_t^2 - \langle X, X\rangle_t$ mart? No!

<u>Def</u> Ito process.

$X_t = X_0 + \underline{\int_0^t b_s \, ds} + \underline{\int_0^t \sigma_s \, dW_s}$, " $E\left[\int_0^t |b_s| \, ds\right] < \infty$ " " $E\left[\int_0^t \sigma_s^2 \, ds\right] < \infty$ "

$\Rightarrow \quad dX_t = b_t \, dt + \sigma_t \, dW_t$

Thm (Ito formula) $f = f(t,x)$, $f \in C^2$, X is a semimart.

Then $f(t, X_t)$ is also a semimartingale and

$$df(t,X_t) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

$$\Rightarrow f(t,X_t) = f(0,X_0) + \int_0^t \partial_t f(s,X_s)\,ds + \int_0^t \partial_x f(s,X_s)\,dX_s + \frac{1}{2}\int_0^t \partial_x^2 f(s,X_s)\,d[X,X]_s$$

Ex $f \in C^2$, X is an Ito process $dX_t = b_t\,dt + \sigma_t\,dW_t$

$\left(X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s\right)$

$$df(t,X_t) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

$$= \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\left(b_t\,dt + \sigma_t\,dW_t\right) + \frac{1}{2}\partial_x^2 f(t,X_t)\,\sigma_t^2\,dt$$

$$= \underbrace{\left(\partial_t f(t,X_t) + b_t\,\partial_x f(t,X_t) + \frac{1}{2}\sigma_t^2\,\partial_x^2 f(t,X_t)\right)}_{=L} dt + \sigma_t \cdot \partial_x f(t,X_t)\,dW_t.$$

① If $L=0$, then $df(t,X_t) = \sigma_t \partial_x f(t,X_t) dW_t$

$$\Rightarrow f(t,X_t) = f(0,X_0) + \int_0^t \sigma_s \partial_x f(s,X_s) dW_s$$

$\Rightarrow f(t,X_t)$ is a mart.

Ex Ito on W_t^2. $f(t,x) = x^2$, $W_t^2 = f(t, W_t)$

$$\partial_t f = 0, \quad \partial_x f = 2x, \quad \partial_x^2 f = 2$$

$$\begin{aligned} df(t,W_t) &= 2W_t dW_t + \tfrac{1}{2}\cdot 2 \cdot d[W,W]_t \\ &= 2W_t dW_t + \tfrac{1}{2}\cdot 2 \cdot dt \\ &= 2W_t dW_t + dt \end{aligned}$$

$$\Rightarrow f(t,W_t) = \underbrace{f(0,W_0)}_{=0} + \int_0^t 2W_s dW_s + t$$

$$\Rightarrow W_t^2 = 2\int_0^t W_s dW_s + t \Rightarrow W_t^2 - t = 2\int_0^t W_s dW_s$$

$M_t = W_t^2 - t, \quad \langle M, M\rangle_t = \int_0^t 4W_s^2\, ds$

$$\mathbb{E}[(W_t^2 - t)^2] = \mathbb{E}\left[\left(\int_0^t 2W_s\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t 4W_s^2\, ds\right]$$

$$= \int_0^t \mathbb{E}[4W_s^2]\, ds = \int_0^t 4s\, ds = 2t^2.$$

<u>Ex</u> Ito on W_t^3, $\quad W_t^3 = f(t, W_t), \quad f(t,x) = x^3$

$\partial_t f(t,x) = 0, \quad \partial_x f(t,x) = 3x^2, \quad \partial_x^2 f(t,x) = 6x$

$$dW_t^3 = 3W_t^2\, dW_t + \frac{1}{2}\cdot 6 \cdot W_t\, dt = 3W_t^2\, dW_t + 3W_t\, dt$$

$$\Rightarrow W_t^3 = \underline{\int_0^t 3W_s^2\, dW_s} + \underline{\int_0^t 3W_s\, ds}$$

$$\langle W^3, W^3\rangle_t = \int_0^t 9W_s^4\, ds.$$

Q. $W_t^6 - \int_0^t 9W_s^4\,ds$ mart?

Q. $\left(W_t^3 - \int_0^t 3W_s\,ds\right)^2 - \int_0^t 9W_s^4\,ds$ mart?

<u>EX</u> $M_t = e^{\lambda W_t - \frac{\lambda^2}{2}t}$ is a mart.

$M_t = f(t, W_t)$, where $f(t,x) = e^{\lambda x - \frac{\lambda^2}{2}t}$

$\partial_t f(t,x) = -\frac{\lambda^2}{2}\cdot f(t,x)$, $\partial_x f(t,x) = \lambda \cdot f(t,x)$, $\partial_x^2 f(t,x) = \lambda^2 f(t,x)$

$$dM_t = -\frac{\lambda^2}{2} f(t, W_t)\,dt + \lambda f(t, W_t)\,dW_t + \frac{1}{2}\lambda^2 f(t, W_t)\,dt$$

$$= \Big(\underbrace{-\frac{\lambda^2}{2} f(t,W_t) + \frac{1}{2}\lambda^2 f(t,W_t)}_{=0}\Big)\,dt + \lambda f(t,W_t)\,dW_t$$

$\Rightarrow M_t = 1 + \int_0^t \lambda \cdot f(s, W_s)\,dW_s \Rightarrow M$ is a mart.

$$\langle M, M \rangle_t = \int_0^t \lambda^2 f^2(s, W_s)\, ds$$

$$= \int_0^t \lambda^2 \cdot e^{2\lambda W_s - \lambda^2 s}\, ds$$

$$\Rightarrow \left(e^{\lambda W_t - \frac{\lambda^2}{2} t}\right)^2 - \int_0^t \lambda^2 e^{2\lambda W_s - \lambda^2 s}\, ds \quad \text{is a mart.}$$

<u>Ex</u>

$$M_t = e^{\lambda\left(\int_0^t s\, dW_s\right) - \frac{\lambda^2 t^3}{6}}, \qquad X_t = \int_0^t s\, dW_s \qquad \Gamma \; dX_t = t\, dW_t$$

$$= e^{\lambda X_t - \frac{\lambda^2 t^3}{6}}$$

$$= f(t, X_t) \quad \text{where } f(t, x) = e^{\lambda x - \frac{\lambda^2 t^3}{6}}$$

$$\partial_t f = -\frac{\lambda^2 t^2}{2} \cdot f \qquad \partial_x f = \lambda \cdot f \qquad \partial_x^2 f = \lambda^2 f$$

$$dM_t = \left(-\frac{\lambda^2 t^2}{2} \cdot f\right) dt + (\lambda \cdot f)\, dX_t + \left(\frac{1}{2}\lambda^2 f\right) d[X, X]_t$$

$$= \left(-\frac{\lambda^2 t^2}{2} f\right) dt + (\lambda \cdot f) \cdot t \cdot dW_t + \left(\frac{1}{2}\lambda^2 f\right) \cdot t^2 dt$$

$$= \lambda \cdot f \cdot t \, dW_t$$

$$\therefore \; M_t = 1 + \int_0^t \lambda \cdot f(s, X_s) \cdot s \, dW_s$$

$$= 1 + \int_0^t \lambda \cdot M_s \cdot s \, dW_s$$

$\Rightarrow$ M is a mart.

$$\therefore \; \mathbb{E}[M_t] = 1$$

$$\Rightarrow \mathbb{E}\left[e^{\lambda X_t - \frac{\lambda^2 t^3}{6}} \right] = 1 \Rightarrow \mathbb{E}\left[e^{\lambda X_t}\right] = e^{\frac{\lambda^2}{2} \cdot \frac{t^3}{3}}$$

$$\Rightarrow X_t \sim N(0, t^3/3).$$

$$\Rightarrow \int_0^t s \, dW_s \sim N(0, t^3/3)$$

<u>Rmk</u> $X_t = \int_0^t f(s) \, dW_s$

then $X_t \sim N\left(0, \int_0^t f^2(s) \, ds\right)$