Last time

Goal → B.S formula
(Price sec in c/s time)

Ito formula

Girsanov Thm

RNM

(better)

Ito integrals & QV.

lim of Ito sums

$P = \{t_0 = 0 < t_1 \cdots < t_n = T\}$

$$\lim_{\|P\| \to 0} \sum_{i=0}^{n-1} D_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) = \int_0^T D_s \, dW_s$$

Always assume $D \to$ adapted process

Notation reminder:

$$\int_0^T D_t \, dt = \text{Rieman Int}$$

$$\int_0^T D_t \, dW_t = \text{Itô int}$$

Theorem 6.17. *If* $\boxed{\boldsymbol{E}\int_0^T D_t^2\,dt < \infty}$ ~~*a.s.*~~*, then:*

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E} I(T)^2 < \infty$.

(2) The process I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I, I]_T = \int_0^T D_t^2\,dt$ *a.s.*

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\Big(\int_0^T D_t\,dW_t\Big)^2 = \boldsymbol{E}\int_0^T D_t^2\,dt$

Itô int.

R-int

Proof.

Last time

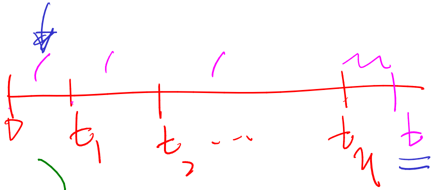
Caution $EX^2 = E(X^2)$ (not $(EX)^2$)

$\int_0^t D_s\,dW_s$ is a mg with qv $\int_0^t D_s^2\,ds$

$\Rightarrow \left(\int_0^t D_s\,dW_s\right)^2 - \int_0^t D_s^2\,ds$ is a mg & take E.

Intuition for Theorem 6.17 (2). Check $I_P(T)$ is a martingale. Last time

Why is $[I,I]_T = \int_0^T D_s^2 \, ds$

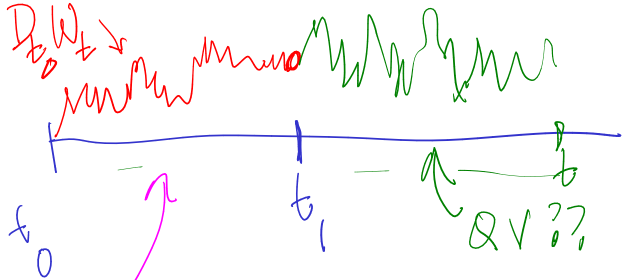


$$I_P(t) = \sum_{0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_t - W_{t_n})$$

$[I_P, I_P]$

(1) Say $0 \le t < t_1$: $I_P(t) = D_{t_0}(W_t - \cancel{W_0})$

I_p



$t \in [t_1, t_2]$

$$I_p(t) = D_{t_0}\left(W_{t_1} - W_{t_0}\right) + D_{t_1}\left(\underline{W_t} - \underline{W_{t_1}}\right)$$

indep

$$[I_p, I_p]_t = D_{t_0}^2\, t$$

$$[I_p, I_p]_{t_1} = D_{t_0}^2\, t_1$$

$$I_p(t) = D_{t_0}^2 (t_1 - t_0) + D_{t_1}^2 (t - t_1)$$

Say $t \in [t_n, t_{n+1})$

Expect $[I_P, I_P]_t = \underbrace{\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_n}^2 (t - t_n)}$

$\|P\| \to 0$

$$\boxed{\int_0^t D_s^2 \, ds} \leftarrow QV \text{ of } \int_0^t D_s \, dW_s$$

Proposition 6.20. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes* ($\alpha, \tilde{\alpha}$ constants, not random)

$$\int_0^T (\alpha D_s + \tilde{\alpha}\tilde{D}_s)\, dW_s = \alpha \int_0^T D_s\, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s\, dW_s$$

Proposition 6.21. $\displaystyle\int_0^{T_1} D_s\, dW_s + \int_{T_1}^{T_2} D_s\, dW_s = \int_0^{T_2} D_s\, dW_s$

Question 6.22. *If $D \geqslant 0$, then must $\int_0^T D_t\, dW_t \geqslant 0$?*

$\boxed{\text{NO}}$

Eg: $D_t = 1$ for all t.

Then $\displaystyle\int_0^T 1\, dW_t = W_T - \cancel{W_0}^{\,0}$ could be ≤ 0.

Remark 6.23. (1) For Riemann-Stieltjes integrals $\frac{d}{dt}\Big(\int_0^t D_r\,dS_r\Big) = \cancel{D_t S_t}$. $D_t \frac{dS_t}{dt}$

(2) For Itô integrals: $\frac{d}{dt}\Big(\int_0^t D_r\,dW_r\Big)$ typically does not exist.

$$\rightarrow \quad \frac{d}{dt}\left(\int_0^t D_s\,ds\right) \overset{\text{F.T.C.}}{=} D_t$$

$$\frac{d}{dt}\left(\int_0^t D_s\,dW_s\right) \quad \text{DNE} \quad \left(\text{Eg: } D = 1, \quad \int_0^t 1\,dW = W_t \leftarrow \text{not diff}\right)$$

6.5. **Semi-martingales and Itô Processes.**

Question 6.24. *What is* $\int_0^t W_s \, dW_s$*?*

IOU techniques to compute

Definition 6.25. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation (aka bounded variation).
- ▷ M is a martingale.

Definition 6.26. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s\, ds$, with $\int_0^t |b_s|\, ds < \infty$ ← (Riemann int)
- ▷ $M_t = \int_0^t \sigma_s\, dW_s$, with $\int_0^t |\sigma_s|^2\, ds < \infty$ ← Itô int.

Remark 6.27. Short hand notation for Itô processes: $\boxed{dX_t = b_t\, dt + \sigma_t\, dW_t.}$

Remark 6.28. Expressing $X = X_0 + B + M$ (or $dX = b\, dt + \sigma\, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s.$$

$$\frac{d}{dt} X_t = \frac{d}{dt} X_0 + \frac{d}{dt}\int_0^t b_s\, ds + \frac{d}{dt}\int_0^t \sigma_s\, dW_s$$

$$\frac{dX_t}{dt} = 0 + b_t + \frac{d}{dt}\int_0^t \sigma_s \, dW_s$$

$$dX_t = b_t \, dt + \cancel{dt}\,\cancel{\frac{d}{dt}}\,\cancel{\int_0^t} \sigma_t \, dW_t$$

R-int Ito int

$$\boxed{dX_t = b_t \, dt + \sigma_t \, dW_t}$$

Notational short hand for

$$\boxed{X_t - X_0 = \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s}$$

Theorem 6.29 (Itô formula)**.** *If $f \in C^{1,2}$, then*

QV.

$$df(t, X_t) = \overbrace{\partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t}^{\text{chain rule.}} + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.30. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Stochastic Calc version of chain rule.

Itô correction,

Proposition 6.31. *If* $X = X_0 + B + M$, *then* $[X, X] = [M, M]$.

I.e. If $X_t - X_0 = \underbrace{\int_0^t b_s \, ds}_{\text{Ignore}} + \underbrace{\int_0^t \sigma_s \, dW_s}$

Then $\boxed{[X, X]_t = \int_0^t \sigma_s^2 \, ds}$

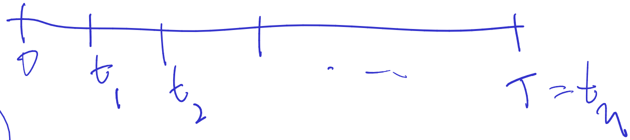
i.e. $\boxed{d[X, X]_t = \sigma_t^2 \, dt = d[M, M]_t}$

Intuition $\quad B \to$ Finite 1^{st} var: $V_{[0,T]} B$ is finite.

$$[X,X]_T = \lim_{\|P\|\to 0} \sum (\Delta_i X)^2 \qquad \Delta_i X = X_{t_{i+1}} - X_{t_i}$$

$[M,M]$.

$$= \lim_{\|P\|\to} \Big(\sum (\Delta_i M)^2 + (\Delta_i B)^2 + 2(\Delta_i M)(\Delta_i B) \Big)$$



Claim: $\lim_{\|P\|\to 0} \sum (\Delta_i B)^2 \leftarrow 0 \leftarrow 1$

$$\lim_{\|P\|\to 0} \sum (\Delta_i B)(\Delta_i M) \to 0 \leftarrow ②$$

Check ①:

$$\sum (\Delta_i B)^2 \le \underbrace{\max_i (\Delta_i B)}_{\substack{\downarrow \|P\|\to 0 \\ 0 \\ (\because B \text{ is cts})}} \underbrace{\left(\sum_i |\Delta_i B|\right)}_{\substack{\downarrow \|P\|\to 0 \\ V_{[0,T]}B \ (\text{finite})}}$$

$$\big\uparrow \lim_{\|P\|\to 0}$$

$$\left(\Rightarrow [B,B]_T = 0\right)$$

Check ② $\lim_{\|P\| \to 0} \sum (\Delta_i B)(\Delta_i M) = 0$

Cauchy-Schwarz

$$\left| \sum (\Delta_i B)(\Delta_i M) \right| \leq \underbrace{\left(\sum (\Delta_i B)^2 \right)^{1/2}}_{\substack{\|P\| \to 0 \\ 0 = [B,B]_T}} \underbrace{\left(\sum (\Delta_i M)^2 \right)^{1/2}}_{\substack{\|P\| \to 0 \\ [M,M]_T}}$$

$\Rightarrow$ ②.

Proposition 6.32 (Uniqueness). *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$, *with:*

- B, C *bounded variation*, $B_0 = C_0 = 0$
- M, N *martingale*, $M_0 = N_0 = 0$.

Then $X_0 = Y_0$, $B = C$ *and* $M = N$.

Use for us:

If $dX = b\,dt + \sigma\,dW_t$

& $dX = \tilde{b}\,dt + \tilde{\sigma}\,dW_t$

then $b = \tilde{b}$

& $\sigma = \tilde{\sigma}$

Intuition :

$$\left.\begin{aligned} X &= X_0 + \underline{\underline{B}} + \underline{\underline{M}} \\ &= Y_0 + \underline{\underline{C}} + \underline{\underline{N}} \end{aligned}\right\} \quad B_0 = M_0 = C_0 = N_0 = 0 \Rightarrow X_0 = Y_0$$

$$\Rightarrow B + M = C + N$$

$$\Rightarrow \underbrace{M - N}_{Mg} = \underbrace{C - B}_{\text{Finite } 1^{st} \text{ var.}} \quad (\Rightarrow 0 \text{ QV})$$

$\Rightarrow M - N$ is a mg & QV of $M - N = 0$ $(\& M_0 - N_0 = 0)$

$$\Rightarrow (M-N)^2 - \underbrace{[M-N, M-N]}_{0} \text{ is a mg}$$

$$\Rightarrow (M-N)^2 \text{ is a mg}$$

$$\Rightarrow E\underbrace{(M_t-N_t)^2}_{} = E(M_0-N_0)^2 = 0$$

$$M_t = N_t \quad \text{a.s.}!! \quad (\Rightarrow B = C).$$

Corollary 6.33. *Let* $dX_t = b_t\,dt + \sigma_t\,dW_t$ *with* $\boldsymbol{E}\int_0^t b_s\,ds < \infty$ *and* $\boldsymbol{E}\int_0^t \sigma_s^2\,ds < \infty$*. Then* X *is a martingale if and only if* $b = 0$*.*

(Assume ...)

$dX = b\,dt + \sigma\,dW$ is mg $\iff$ $b = 0$

Check: Say X is a mg.

Then $X_t = X_0 + \underbrace{\int_0^t b_s\,ds}_{\text{fin } 1^{st} \text{ var}} + \underbrace{\int_0^t \sigma_s\,dW_s}_{\text{Mg}}$

$= X_0 + 0 + (X_t - X_0)$

Uniq $\Rightarrow$ $b = 0$!!

Definition 6.34. If $dX = b\,dt + \sigma\,dW$, define $\int_0^T D_t\,dX_t = \int_0^T D_t b_t\,dt + \int_0^T D_t \sigma_t\,dW_t$.

Remark 6.35. Note $\int_0^T D_t b_t\,dt$ is a *Riemann integral*, and $\int_0^T D_t \sigma_t\,dW_t$ is a *Itô integral*.

6.6. **Itô's formula.**

Remark 6.36. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$$

$$\frac{df(t,X_t)}{\cancel{dt}} = \partial_t f(t,X_t) \cdot \frac{dt}{\cancel{dt}} + \partial_x f(t,X_t)\, \frac{dX_t}{\cancel{dt}}$$

$f \in C^{1,2}$ means: $\partial_t f$ exists & is cts

$\partial_x f$ & $\partial_x^2 f$ exist & are cts.

Same terms from Chain Rule.

Theorem (Itô's formula, Theorem 6.29). *If* $\boxed{f \in C^{1,2}}$, *then*

$$df(t,X_t) = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

Remark 6.37. If $dX_t = b_t\,dt + \sigma_t\,dW_t$ then

$$df(t,X_t) = \left(\partial_t f(t,X_t) + b_t \partial_x f + \frac{1}{2}\sigma_t^2 \partial_x^2 f\right)dt + \partial_x f(t,X_t)\sigma_t\,dW_t\,.$$

$$d f(t,X_t) = \partial_t f\,dt + \partial_x f\,\underbrace{dX_t}_{b\,dt + \sigma\,dW} + \frac{1}{2}\partial_x^2 f\,\underbrace{d[X,X]}_{\sigma_t^2\,dt}$$

Example 6.38. $\boxed{\text{Find the quadratic variation of } W_t^2.}$

Choose $f(t,x) = x^2$

$\partial_t f = 0$

$\partial_x f = 2x$

$\partial_x^2 f = 2$

$$d f(t, W_t) = d(W_t^2)$$

$$= \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, d[W, W]$$

$$\Rightarrow d(W_t^2) = 2W_t \, dW_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$\Rightarrow \boxed{d(W_t^2) = 2W_t\,dW_t + dt}$$

$$\Rightarrow d[W^2, W^2]_t = 4W_t^2\,dt\,,$$