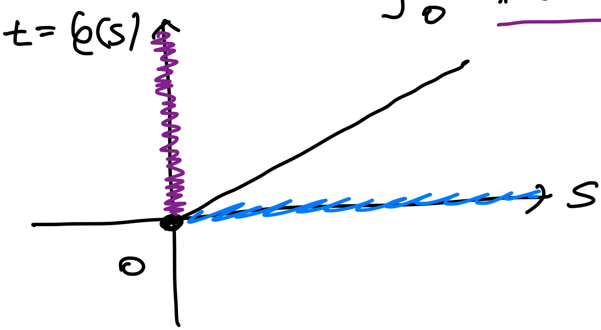
3 (b) If $X \geq 0$, $\underline{\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)\,dt}$

$\varphi(x) \geq 0$.

$$\mathbb{E}[\varphi(X)] = \int_0^\infty \mathbb{P}(\varphi(X) \geq t)\, dt$$

$$\overset{\varphi(s)=t}{=} \int_0^\infty \underbrace{\mathbb{P}(\varphi(X) \geq \varphi(s))}_{①}\, \varphi'(s)\,ds$$




$$= \int_0^\infty \overset{②}{\mathbb{P}(X \geq s)}\, \varphi'(s)\,ds \qquad \blacksquare$$

① = ②

3. (c) Counterexample!

3. (a) $X \geq 0$. $X \sim p$

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)\, dt$$

pf $\mathbb{P}(X \geq t) = \int_t^\infty p(x)\, dx$

$$\int_0^\infty \mathbb{P}(X \geq t)\, dt = \int_0^\infty \left[ \int_t^\infty p(x)\, dx \right] dt$$

$$= \int_0^\infty \left[ \int_0^\infty \mathbb{1}_{(t \leq x)}\, p(x)\, dx \right] dt$$

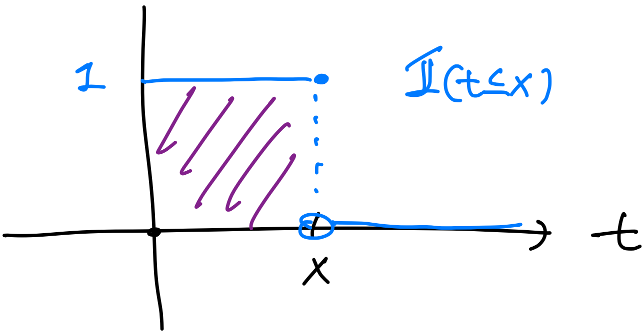
✓

$$\mathbb{1}_{(t \leq x)} = \begin{cases} 1 & t \leq x \\ 0 & t > x \end{cases}$$

Fubini

$$= \int_0^\infty \int_0^\infty \mathbb{1}_{(t \leq x)}\, p(x)\, dt\, dx$$

$$= \int_0^{\infty} p(x) \underbrace{\int_0^{\infty} \mathbb{1}(t \le x)\, dt}_{= x}\, dx$$




$$= \int_0^{\infty} p(x) \cdot x \, dx = \mathbb{E}[X].$$

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1 (a) $V_n$ : the price of security at time $n$

$$V_N = g(S_N)$$

$$V_{N-1} = \frac{1}{1+r} \tilde{\mathbb{E}}_{N-1}[V_N]$$

$$= \frac{1}{1+r} \tilde{\mathbb{E}}_{N-1}[g(S_N)]$$

$S_{n+1} = S_n \cdot X_{n+1}$ where $X_{n+1} = \begin{cases} u & \omega_{n+1} = H \\ d & \omega_{n+1} = T \end{cases}$

$$= \frac{1}{1+r} \tilde{\mathbb{E}}_{N-1}\left[ g(S_{N-1} \cdot X_N) \right]$$

$$= \frac{1}{1+r} h(S_{N-1})$$

where $\quad h(x) = \tilde{\mathbb{E}}\left[ g(x \cdot X_N) \right]$

$$= g(x \cdot u) \cdot \tilde{p} + g(x \cdot d) \cdot \tilde{q}$$

$$= \frac{1}{1+r} \left[ g(u \cdot S_{N-1}) \cdot \tilde{p} + g(d \cdot S_{N-1}) \cdot \tilde{q} \right]$$

$$\therefore \quad V_{N-1} = \frac{1}{1+r} \left[ g(u \cdot S_{N-1}) \cdot \tilde{p} + g(d \cdot S_{N-1}) \cdot \tilde{q} \right]$$

$$= f_{N-1}(S_{N-1})$$

$$f_{N-1}(x) = \frac{1}{1+r} \left[ g(ux) \tilde{p} + g(dx) \tilde{q} \right].$$

$\rightsquigarrow$ "formula for $f_n$".

$$V_{N-2} = \frac{1}{1+r} \tilde{\mathbb{E}}_{N-2}[V_{N-1}]$$

$$= \frac{1}{1+r} \tilde{\mathbb{E}}_{N-2}[f_{N-1}(S_{N-1})]$$

$$= \cdots = \text{"} \qquad \text{"}$$

---

5. (a) $Y \sim N(0,1)$, $g(x) = \mathbb{E}[(e^{x+Y} - k)_+]$

$$g(x) = \mathbb{E}\left[ (e^{x+Y} - k) \cdot \mathbb{1}(e^{x+Y} - k \geq 0) \right]$$

$$x_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} = x \cdot \mathbb{1}(x \geq 0)$$

$$= \int_{-\infty}^{\infty} (e^{x+y} - k) \cdot \mathbb{1}(e^{x+y} - k \geq 0) \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \, dy$$

$$= \int_{-\infty}^{\infty} \left(e^{x+y} - k\right) \cdot \mathbb{1}_{(y \geq -x+\log k)} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$e^{x+y} - k \geq 0 \iff y \geq -x + \log k$$

$$= \int_{-x+\log k}^{\infty} \left(e^{x+y} - k\right) \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$\int_{-\infty}^{\infty} = \underbrace{\int_{-x+\log k}^{\infty}} + \underbrace{\int_{-\infty}^{-x+\log k}}$$

$$= \underbrace{\int_{-x+\log k}^{\infty} e^{x+y} \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy}_{= \# \cdot N(\triangle)} - k \cdot \underbrace{\int_{-x+\log k}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy}_{= N(x - \log k)}$$

5.(b) $X, Y \sim N(0,1)$. $X, Y$ independent

$$\mathbb{E}\left[ (e^{X+Y} - k)_+ \,\middle|\, X \right]$$

$$= \mathbb{E}\left[ (e^{X+Y} - k)_+ \,\middle|\, \sigma(X) \right]$$

Ind. Lem.

$$= g(X) \quad \text{where}$$

$$g(x) = \mathbb{E}\left[ (e^{x+Y} - k)_+ \right]$$

$$\mathbb{E}\left[ (e^{X+Y} - k)_+ \,\middle|\, X \right](\omega) = g(X(\omega))$$

4 (a) $W_t = \underbrace{W_t - W_s}_{X} + \underbrace{W_s}_{Y}$

$$\mathbb{E}_s[W_t^3] = \mathbb{E}_s[(X+Y)^3]$$

we need $\underbrace{\mathbb{E}_s[X^3]}_{=\mathbb{E}[X^3]}$, $\underbrace{\mathbb{E}_s[X^2]}_{=\mathbb{E}[X^2]}$, $\underbrace{\mathbb{E}_s[X]}_{=\mathbb{E}[X]}$

$\mathbb{E}_s[Y^3]$, $\mathbb{E}_s[Y^2]$, $\mathbb{E}_s[Y]$

$$\int_{-\infty}^{\infty} x^3 \cdot \frac{e^{\wedge \frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \, dx = 0.$$

2.(a) $\lambda \cdot \mathbb{1}(x \geq \lambda) \leq |X|$, $\lambda > 0$

$\Rightarrow \lambda^p \cdot \mathbb{1}(x \geq \lambda) \leq |X|^p$, $p > 0$

$$V_0 = \star \left( V_1(1) + V_1(2) \right)$$

$$V_2(\star) \quad V_2(\star) = V_2(\star) \quad V_2(\star)$$