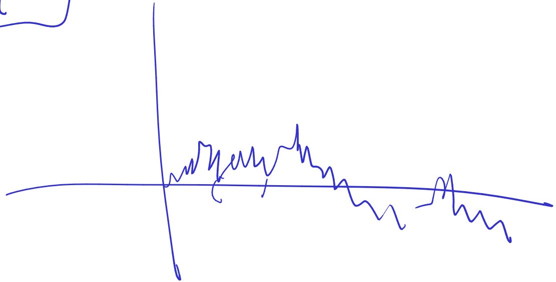
Reminder : O.H. Today 3:30 & Tomorrow 12:00 (Zoom)

Last time : $\boxed{\text{Brownian Motion}}$

Cts time RW



W_t = BM at time t

(1) $W_t - W_s \sim N(0, t-s)$

(2) $W_t - W_s$ is indep of $\mathcal{F}_s$

5.4. **Martingales.**

Definition 5.11. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.12. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Disc time:

$$E_n M_{n+1} = M_n$$

Proposition 5.13. *Brownian motion is a martingale.*

Proof.

Want to show $E_s W_t = W_s$

$$E_s W_t = E_s\left(W_t - W_s + W_s\right) = \underbrace{E_s(W_t - W_s)}_{\text{Indep}} + \underbrace{E_s W_s}_{\mathcal{F}_s \text{ meas}}$$

$$= \underbrace{E(W_t - W_s)}_{0} + W_s$$

$$= W_s.$$

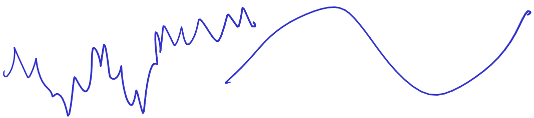
6. **Stochastic Integration**

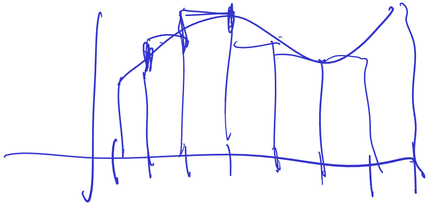
Partition

6.1. **Motivation.**

- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$

- Net gain/loss from changes in stock price: $\displaystyle\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  - ▷ $\|P\| = \max_k (t_{k+1} - t_k)$. ← Mesh size of P
  - ▷ Riemann-Stieltjes integral: $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t$,
  - ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  - ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.





6.2. First Variation.

$\Sigma |X_{t_{k+1}} - X_{t_k}| = X_n - X_0$

Definition 6.1. For any process X, define the $\boxed{\textit{first variation}}$ by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|\,.$$

Remark 6.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]}X < \infty$.

Proposition 6.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

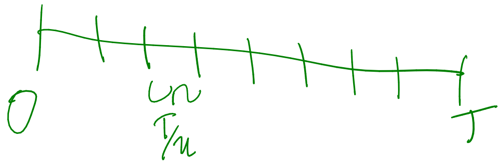
Remark 6.4. In fact, $V_{[0,T]} W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 6.5. The Riemann-Stieltjes integral $\int_0^T b_t \, dW_t$ *does not exist.*

$\Delta_k W = W_{t_{k+1}} - W_{t_k}$

$$\to E\, V_{[0,T]} W = \lim_{\|P\|\to 0} E \sum |\Delta_k W|$$

Say $P \to$ "uniform"



I.e. say $t_{k+1} - t_k = \frac{T}{n}$

$$E|\Delta_k W| = E\left|W_{t_{k+1}} - W_{t_k}\right| = E\left|N(0, t_{k+1} - t_k)\right|$$

$$= E\left|N\left(0, \frac{T}{n}\right)\right| = E\left|\sqrt{\frac{T}{n}}\, N(0,1)\right|$$

$$= \sqrt{\frac{T}{n}}\, \underbrace{E|N(0,1)|}_{\text{some finite constant.}}$$

ต่อ $$E\, V_{[0,T]} W = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} E\,|\Delta_k W|$$

$$= \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} \sqrt{\frac{T}{n}}\; E\,|N(0,1)|$$

$$= \lim_{\|P\|\to 0} \sqrt{nT}\; E\,|N(0,1)| \longrightarrow \infty$$

6.3. **Quadratic Variation.**

$$V_T M = \lim_{\|P\|\to 0} \sum |M_{t_{k+1}} - M_{t_k}|$$

Definition 6.6. If M is a continuous time adapted process, define

$$[M,M]_T = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 \,.$$

Proposition 6.7. *For continuous processes the following hold:*

(1) *Finite first variation implies the quadratic variation is 0.*

(2) *Finite (non-zero) quadratic variation implies the first variation is infinite.*

Revisit this later (Important)

Proposition 6.8. $[W,W]_T = T$ *almost surely.*

Remark 6.9. For use in the proof: $\operatorname{Var}(\mathcal{N}(0,\sigma^2)^2) = \boldsymbol{E}\mathcal{N}(0,\sigma^2)^4 - (\boldsymbol{E}\mathcal{N}(0,\sigma^2)^2)^2 = 2\sigma^4.$

Proof:.

$$3\sigma^4 - \sigma^4$$

$$\boxed{E\,\mathcal{N}(0,\sigma^2)^2 = \sigma^2}$$

$$[W,W]_T = \lim_{\|P\|\to 0} \sum (\Delta_k W)^2$$

Assume uniform mesh

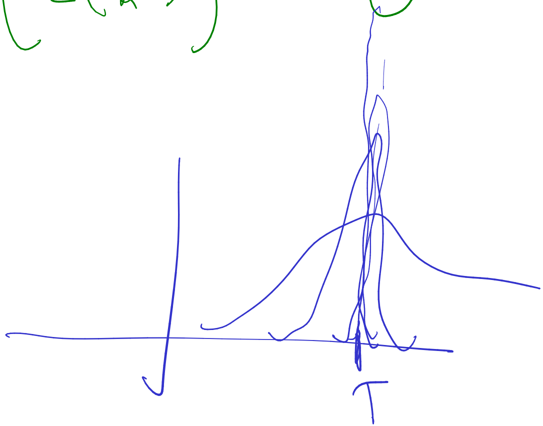
NTS $= T$ a.s.

Will show ① $\lim\limits_{\|P\| \to 0} E \sum (\Delta_k W)^2 = \underline{\underline{T}}$

& ② $\lim\limits_{\|P\| \to 0} \operatorname{Var}\left(\sum (\Delta_k W)^2 \right) = 0$

Check ①:

$$E \sum (\Delta_k W)^2 = \sum E\, N(0, t_{k+1} - t_k)^2$$



$$= \sum (t_{k+1} - t_k) = T$$

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② $$\mathrm{Var}\left(\sum_{k=0}^{n-1} (\Delta_k W)^2\right) = \sum_{k=0}^{n-1} \mathrm{Var}\left((\Delta_k W)^2\right) \quad (\text{by indep})$$

$$= \sum_{k=0}^{n-1} \mathrm{Var}\left(N(0, t_{k+1} - t_k)^2 \right)$$

$$= \sum_{k=0}^{n-1} \mathrm{Var}\left(N\left(0, \frac{T}{n}\right)^2 \right) = \sum_{k=0}^{n-1} 2 \frac{T^2}{n^2}$$

$$= \frac{2T^2}{n} \xrightarrow[(n \to \infty).]{\|P\| \to 0} 0$$

$$\boxed{[W, W]_t = t \quad \text{a.s.}}$$

Proposition 6.10. *$W_t^2 - [W,W]_t$ is a martingale.*

Check : $E_s\left(W_t^2 - [W,W]_t\right) \overset{\text{Want}}{=} W_s^2 - [W,W]_s$

$$E_s\left(W_t^2 - [W,W]_t\right) = E_s\left(W_t^2 - t\right)$$

$$= E_s W_t^2 - t$$

$$= E_s\left(W_t - W_s + W_s\right)^2 - t$$

$$= E_s\left[(W_t - W_s)^2 + W_s^2 + 2(W_t - W_s)W_s\right] - t$$

$$\overbrace{\mathbb{E}_s}^{\text{means}}$$

$$\left(W_t - W_s \sim N(0, t-s)\right) \quad = E(W_t - W_s)^2 + W_s^2 + 2E_s\left[(W_t - W_s)W_s\right] - t$$

$$= \cancel{t} - s + W_s^2 + 2W_s \underbrace{E_s(W_t - W_s)}_{0} - \cancel{t}$$

$$= W_s^2 - s = W_s^2 - [W,W]_s.$$

$$\boxed{\Rightarrow W^2 - [W,W] \text{ is a } M_g.}$$

Theorem 6.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E} M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M, M]_t < \infty$.

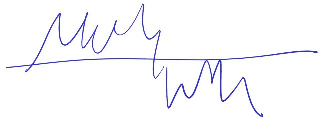
(2) *In this case* $M_t^2 - [M, M]_t$ *is a continuous martingale.*

(3) *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$*, then we must have* $A_t = [M, M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

Remark 6.13. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 6.14. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.



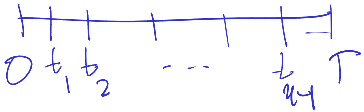
Say X is diff.

$$X_{t+h} - X_t \approx \left(\frac{dX}{dt}\right) \cdot h$$

Finite QV: $|X_{t+\delta t} - X_t|^2 \approx \delta t$

6.4. **Itô Integrals.**

- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i t_{i+1} - t_i$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$



$I_P(T)$ = "Itô sum"

Definition 6.15. The *Itô Integral* of D with respect to Brownian motion is defined by

$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) \, .$$

Remark 6.16. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

$\int_0^T D_t^2\,dt \to$ R. int

Theorem 6.17. *If* $\boldsymbol{E}\int_0^T D_t^2\,dt < \infty$ *a.s., then:*

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E} I(T)^2 < \infty$.

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I, I]_T = \int_0^T D_t^2\,dt$ *a.s.*

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

$$I_T = \int_0^T D_t\,dW_t$$

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\Big(\underbrace{\int_0^T D_t\,dW_t}_{\text{Ito int.}}\Big)^2 = \boldsymbol{E}\underbrace{\int_0^T D_t^2\,dt}_{\text{Riemann Int}}$

Proof.

Intuition ① $I_T = \int_0^T D_s\,dW_s$

Thm $\Rightarrow$ $\underline{I}$ is a mg & $[I,I]_T = \int_0^T D_s^2\,ds.$

② Know $I_t^2 - [I,I]_t$ is a mg.

(3) $\Rightarrow E\left(I_t^2 - [I,I]_t\right) = E_0\left(I_t^2 - [I,I]_t\right)$

$$= I_0^2 - [I,I]_0 = 0$$

(4) $\Rightarrow E I_t^2 = E[I,I]_t$

$$= E\int_0^t D_s^2\, ds \qquad (\text{by } (1))$$

$\Rightarrow$ Itô isom.

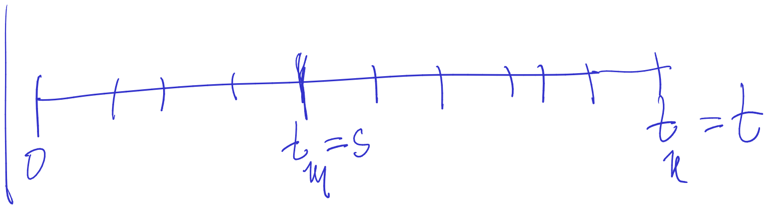
Intuition why Itô int is a mg?

Simplest case: Check $I_p(t)$ is a mg in a simple case.

Compute $E_s I_p(t) \overset{\text{Want}}{=} I_p(s)$

Say $s = t_m$

$t = t_n$ $\quad m < n$



$$I_p(s) = \sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$E_s I_p(t) = E_s \sum_{k=0}^{n-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) \qquad (s = t_m)$$

$$= \underbrace{E_s \sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)}_{E_s \ \text{meas} \ (s = t_m)} + E_s \sum_{k=m}^{n-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$=$$

$$= \underbrace{\sum_{k=0}^{m-1} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)}_{I_p(s)} + \sum_{k=m}^{n-1} E_s D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s E_{t_k} D_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) \quad (\text{tower})$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s\left[D_{t_k} E_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right)\right] \quad (\because D_{t_k} \text{ is } \mathcal{F}_{t_k} \text{ meas})$$

$$= I_p(s) + \sum_{k=m}^{n-1} E_s\Big[D_{t_k} \underbrace{E\left(W_{t_{k+1}} - W_{t_k}\right)}_{0} \Big] \quad \text{(indep)}$$

$$= I_p(s)$$

$$\boxed{\Rightarrow \quad E_s I_p(t) = I_p(s)}$$