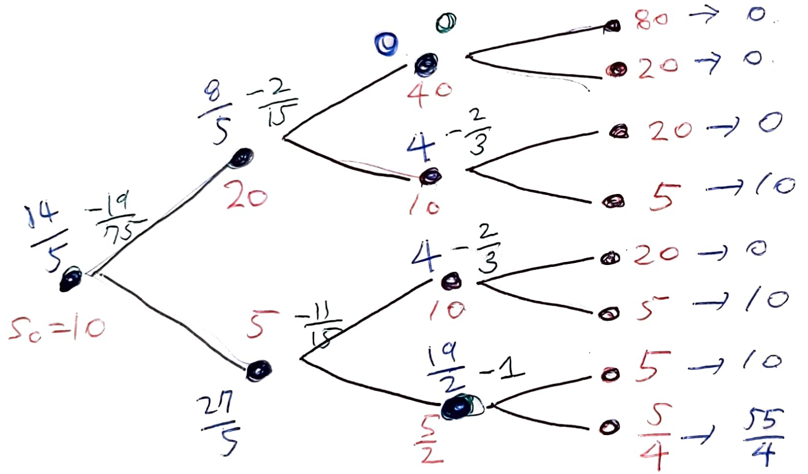
pricing and Hedging security

Binomial Model: $N=3$, $r=\frac{1}{4}$, $p=\frac{9}{10}$, $q=\frac{1}{10}$, $S_0=10$, $u=2$, $d=\frac{1}{2}$.

price and Hedge $(15-S_3)^+ = \begin{cases} 15-S_3 & 15>S_3 \\ 0 & 15\le S_3 \end{cases}$

- RN $\tilde{p}$, $\tilde{q}$ $\quad u\tilde{p}+d\tilde{q}=1+r$, $\tilde{p}+\tilde{q}=1 \Rightarrow \tilde{p}=\tilde{q}=\frac{1}{2}$



$$V_n = \frac{1}{(1+r)^{N-n}}\tilde{E}_n[V_N], \quad V_2 = \frac{1}{1+r}\tilde{E}_2[V_3]$$

$$V_2(T,T,*) = \frac{4}{5}\cdot\left(\frac{1}{2}\cdot 10+\frac{1}{2}\cdot\frac{55}{4}\right) = \frac{4}{5}\left(5+\frac{55}{8}\right) = 4\left(1+\frac{11}{8}\right) = \frac{4\cdot 19}{8} = \frac{19}{2}$$

$$V_1(T,*,*) = \frac{4}{5}\cdot\left(\frac{1}{2}\cdot 4+\frac{1}{2}\cdot\frac{19}{2}\right) = \frac{27}{5}$$

$$\Delta_n = \frac{V_{n+1}(\omega_{n+1}=H) - V_{n+1}(\omega_{n+1}=T)}{(u-d)S_n} = \frac{V_{n+1}(\omega_{n+1}=H) - V_{n+1}(\omega_{n+1}=T)}{S_{n+1}(\omega_{n+1}=H) - S_{n+1}(\omega_{n+1}=T)}$$

$$\Delta_2(T,T,*) = \frac{10 - \frac{55}{4}}{5 - \frac{5}{4}} = \frac{40-55}{20-5} = -1$$

$$\Delta_1(H,*) = \frac{-4}{30} = -\frac{2}{15}$$

$$\Delta_1(T,*) = \frac{4 - \frac{19}{2}}{10 - \frac{5}{2}} = \frac{8-19}{20-5} = \frac{-11}{15}$$

$$\Delta_0 = \frac{\frac{8}{5} - \frac{27}{5}}{20-5} = \frac{-19}{75}$$

Conditional Expectation

<u>Ex</u> $\mathbb{E}_t[X]$ is the best "least square" approximation of X among $\mathcal{F}_t$-measurable r.v.

<u>pf</u> $\forall Y$: $\mathcal{F}_t$-m'ble, $\mathbb{E}[(X-Y)^2] \geq \mathbb{E}[(X-\mathbb{E}_t[X])^2]$

$$\mathbb{E}[(X-Y)^2] = \mathbb{E}[(\underbrace{X-\mathbb{E}_t[X]}_{a} + \underbrace{\mathbb{E}_t[X]-Y}_{b})^2]$$

$$= \underline{\mathbb{E}[(X-\mathbb{E}_t[X])^2]} + 2\cdot \underbrace{\mathbb{E}[(X-\mathbb{E}_t[X])\cdot(\mathbb{E}_t[X]-Y)]}_{=I} + \underbrace{\mathbb{E}[(\mathbb{E}_t[X]-Y)^2]}_{\geq 0}$$

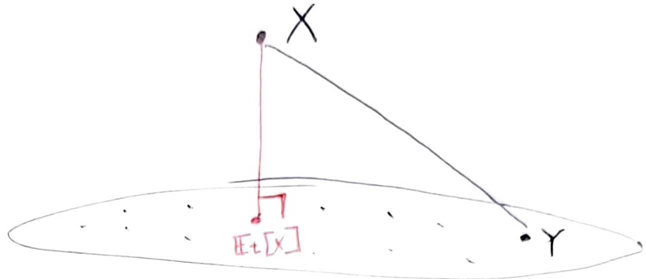
$$I = \mathbb{E}[\mathbb{E}_t[(X-\mathbb{E}_t[X])(\overbrace{\mathbb{E}_t[X]-Y}^{\mathcal{F}_t\text{-m'ble}})]]$$

$$= \mathbb{E}[(\mathbb{E}_t[X]-Y)\cdot \mathbb{E}_t[X-\mathbb{E}_t[X]]]$$

$$= \mathbb{E}[(\mathbb{E}_t[X]-Y)\cdot(\mathbb{E}_t[X] - \underbrace{\mathbb{E}_t[\mathbb{E}_t[X]]}_{\mathbb{E}_t[X]})] = 0.$$

$$\therefore \mathbb{E}[(X-Y)^2] \geq \mathbb{E}[(X-\mathbb{E}_t[X])^2]$$

∎



$\mathcal{H} = \{\mathcal{F}_t\text{-m'ble r.v.}\}$

Ex $\quad 0 \le s \le t, \quad \mathbb{E}_s[W_t^2 - t] = W_s^2 - s.$

$$\mathbb{E}_s[W_t^2 - t] = \mathbb{E}_s\Big[(\underbrace{W_t - W_s}_{a} + \underbrace{W_s}_{b})^2 - t\Big]$$

$$= \mathbb{E}_s\big[(W_t - W_s)^2 + 2(W_t - W_s)\cdot W_s + W_s^2 - t\big]$$

$$= \mathbb{E}_s\big[(W_t - W_s)^2\big] + 2\cdot\mathbb{E}_s\big[(W_t - W_s)\cdot W_s\big] + \mathbb{E}_s\big[W_s^2 - t\big]$$

$$= \mathbb{E}_s\big[(W_t - W_s)^2\big] + 2W_s\cdot\mathbb{E}_s\big[W_t - W_s\big] + W_s^2 - t$$

$W_t - W_s \sim N(0, t-s)$

$$= \mathbb{E}\big[(W_t - W_s)^2\big] + 2W_s\cdot\mathbb{E}\big[W_t - W_s\big] + W_s^2 - t$$

$$= t - s + 2\cdot W_s\cdot 0 + W_s^2 - t = W_s^2 - s. \qquad \blacksquare$$

Def X is a Gaussian r.v. (normal r.v.) with mean μ, variance σ^2,

$X \sim N(\mu, \sigma^2)$. $\quad \mathbb{P}(a \le X \le b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

Def $(X_1, X_2, \cdots, X_n)$ is a Gaussian Vector if

$\forall \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$, $\lambda_1 X_1 + \cdots + \lambda_n X_n$ is a Gaussian r.v.

prop Each of $X_1, X_2, \cdots, X_n$ is a Gaussian r.v. and they are independent

Then. $(X_1, \cdots, X_n)$ is a Gaussian vector.

EX X, Y, Z i.i.d $N(0,1) \Rightarrow (X, Y, Z)$ is a Gaussian vector

$\Rightarrow X+Y,\ X-Y+3Z, \cdots$ are Gaussian r.v.

$\Rightarrow (X+Y, X-Y)$ is a Gaussian vector

$\lambda_1(X+Y) + \lambda_2(X-Y) = (\lambda_1+\lambda_2)X + (\lambda_1-\lambda_2) \cdot Y$

prop $(X_1, X_2, \cdots, X_n)$ is a Gaussian vector. Then

X_i, X_j Independent $\Longleftrightarrow \mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right] = 0$

prop $X \sim N(\mu, \sigma^2)$, $\mathbb{E}[e^{tX}] = e^{t\mu + \frac{t^2\sigma^2}{2}}$

pf $\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \cdots = e^{t\mu + \frac{t^2\sigma^2}{2}}$ ∎

prop (Ind Lem, Discrete) X: Independent of $\mathcal{F}_n$

Y: $\mathcal{F}_n$-m'ble

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}_n[f(X,Y)] = g(Y) \quad \text{where} \quad g(y) = \mathbb{E}[f(X,y)]$$

prop (Ind Lem, Continuous) X: Ind of $\mathcal{F}_t$

Y: $\mathcal{F}_t$-m'ble

$f: \mathbb{R}^2 \to \mathbb{R}$

$g(y) = y^2$

$g(W_t) = W_t^2$

$$\mathbb{E}_t[f(X,Y)] = g(Y) \quad \text{where} \quad g(y) = \mathbb{E}[f(X,y)]$$

Rmk $\mathbb{E}_t[f(X,Y)] \approx \mathbb{E}_t[f(X,y)] = \mathbb{E}[f(X,y)] = g(y)$

Rmk $\mathbb{E}_t[X] = \mathbb{E}[X \mid \mathcal{F}_t]$

$\mathcal{F}_t = \{\text{events}\}$

$\mathcal{F} = \{\text{events}\}$ $\rightarrow$ $\mathbb{E}[X \mid \mathcal{F}]$

* X is $\mathcal{F}$-m'ble $\Rightarrow$ $\mathbb{E}[X \mid \mathcal{F}] = X$.
* X is ind of $\mathcal{F}$ $\Rightarrow$ $\mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X]$.

Ex Y is a random r.v.

$\sigma(Y) = \{\text{events from observations of } Y\}$

$= \{ \{Y \in B\} \mid B \subseteq \mathbb{R} \}$.

$\mathbb{E}[X \mid \sigma(Y)]$

$\underline{\text{EX}}$ $X \sim N(0,1)$, $\mathcal{E} = \begin{cases} +1 & \text{with prob } \frac{1}{2} \\ -1 & \text{with prob } \frac{1}{2} \end{cases}$. $X, \mathcal{E}$ are independent.

- $\mathbb{E}\left[e^{t\mathcal{E}X} \mid \sigma(\mathcal{E}) \right] = g(\mathcal{E})$, $\quad g(y) = \mathbb{E}\left[e^{tyX} \right]$
- $\mathbb{E}\left[e^{t\mathcal{E}X} \mid \sigma(X) \right] = f(X)$, $\quad f(x) = \mathbb{E}\left[e^{t\mathcal{E}x} \right]$

$$g(y) = \mathbb{E}\left[e^{ty \cdot X} \right] = e^{\frac{t^2 y^2}{2}} \Big/ f(x) = \frac{e^{tx} + e^{-tx}}{2}$$

- $\mathbb{E}\left[e^{t\mathcal{E}X} \mid \sigma(\mathcal{E}) \right] = e^{\frac{t^2 \mathcal{E}^2}{2}} = e^{\frac{t^2}{2}}$
- $\mathbb{E}\left[e^{t\mathcal{E}X} \mid \sigma(X) \right] = \dfrac{e^{tX} + e^{-tX}}{2}$

$$\mathbb{E}\left[e^{t\mathcal{E}X} \right] = \mathbb{E}\left[\mathbb{E}\left[e^{t\mathcal{E}X} \mid \sigma(\mathcal{E}) \right] \right] = \mathbb{E}\left[e^{\frac{t^2}{2}} \right] = e^{\frac{t^2}{2}}$$

$Y = \varepsilon X$, $\quad \mathbb{E}[e^{t\varepsilon X}] = e^{t^2/2}$ = MGF of $N(0,1)$ at t

$= \mathbb{E}[e^{tY}]$

= MGF of Y at t.

$$\mathbb{P}(a \le \varepsilon X \le b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

<u>EX</u> X, Y i.i.d $N(0,1)$

* $X+Y$, $X-Y$ are independent

<u>pf</u> $(X+Y, X-Y)$: Gaussian vector.

So, $\mathbb{E}[(X+Y)\cdot(X-Y)] = 0$

* $X+Y$, $X-Y$ are NOT independent if they are conditioned on $\sigma(X)$.

$\underline{\text{pf}}$ $\mathbb{E}[(X+Y)(X-Y) \mid \sigma(X)] \neq \mathbb{E}[X+Y \mid \sigma(X)]\, \mathbb{E}[X-Y \mid \sigma(X)]$

$$\mathbb{E}[X^2 - Y^2 \mid \sigma(X)] = \mathbb{E}[X^2 \mid \sigma(X)] - \mathbb{E}[Y^2 \mid \sigma(X)]$$
$$= X^2 - \mathbb{E}[Y^2] = X^2 - 1$$

$$\mathbb{E}[X \pm Y \mid \sigma(X)] = \mathbb{E}[X \mid \sigma(X)] \pm \mathbb{E}[Y \mid \sigma(X)]$$
$$= X \pm \mathbb{E}[Y] = X$$

$\underline{\text{Ex}}$ $X_1, X_2, \cdots, X_n$ i.i.d $\quad S_n = X_1 + X_2 + \cdots + X_n$

$$\mathbb{E}[X_1 \mid S_n] = \mathbb{E}[X_1 \mid \sigma(S_n)]$$

$\underline{\text{pf}}$ $\mathbb{E}[X_i \mid S_n] = \mathbb{E}[X_j \mid S_n]$ $\quad \forall_{i,j}$

$$\sum_{i=1}^{n} \mathbb{E}[X_i \mid S_n] = \mathbb{E}\left[\sum_{i=1}^{n} X_i \mid S_n\right]$$

$$= \mathbb{E}[S_n \mid S_n] = S_n$$

$$n\,\mathbb{E}[X_1 \mid S_n] = S_n \Rightarrow \mathbb{E}[X_1 \mid S_n] = \frac{S_n}{n} \qquad \blacksquare$$