Last time: RNM

Equivalent meas. $\tilde{P}$ under which

$$\tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$$

(Disc stock is a $\tilde{P}$ - mg).

Binom model:

Coin flip $\xrightarrow{p_1}$ Heads, $\xrightarrow{q_1}$ tails

RNM:

Coin flip $\xrightarrow{\text{Heads}}$ $\tilde{p}_1$, $\xrightarrow{\text{tails}}$ $\tilde{q}_1$

$$\boxed{\tilde{p}_1 = \frac{(1+r) - d}{u - d}}$$

last time

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)}$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Proof of Proposition 4.1.

Security Pays V_N at time N

AFP at time $n \le N$ is

RNP formula

$$\boxed{V_n = \frac{1}{D_n} \tilde{E}_n \left(D_N V_N \right)}$$

D_n = Discount factor $= \dfrac{1}{(1+r)^n}$

Pf: Want a R. Pf.

$\iff$ Want a self fin port $\Rightarrow$ $\boxed{X_N = V_N.}$

$\Updownarrow$

Disc wealth is a $\tilde{P}$ - mg.

Choose $X_n = \dfrac{1}{D_n} \tilde{E}_n(D_N X_N)$ & $\boxed{X_N = V_N.}$

Know $X_N = V_N$ (by choice)

$$\boxed{\Rightarrow D_{n+1} X_{n+1} = \tilde{E}_{n+1}(D_N X_N)}$$

NTS $D_n X_n$ is a $\tilde{P}$ - mg

↳ Check: $D_n X_n = \tilde{E}_n \left(D_{n+1} X_{n+1} \right)$

Compute $\tilde{E}_n \left(D_{n+1} X_{n+1} \right) = \tilde{E}_n \left(\tilde{E}_{n+1} \left(D_N X_N \right) \right)$

$$\overset{\text{Tower}}{=} \tilde{E}_n \left(D_N X_N \right)$$

$$= D_n X_n$$!! Done!!

Example 4.59. Consider two stocks S^1 and S^2, $u = 2$, $d = 1/2$.

- The coin flips for S^1 are heads with probability 90%, and tails with probability 10%.
- The coin flips for S^2 are heads with probability 99%, and tails with probability 1%.
- Which stock do you like more?
- Amongst a call option for the two stocks with strike K and maturity N, which one will be priced higher?

Both call options are worth exactly the same

[RNM only depends on u, d, r & N
does not depend on p & q]

Remark 4.60. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$. → $\tilde{p}_1$ & $\tilde{q}_1$ Do not depend on p_1 & q_1

Concepts that will be generalized to continuous time.

- Probability measure: Lebesgue integral, and not a finite sum. Same properties.
- Filtration: Same intuition. No easy description.
- Conditional expectation: Same properties, no formula.
- Risk neutral measure: Formula for $\tilde{\boldsymbol{P}}$ is complicated (Girsanov theorem.)
- Everything still works because of of Theorem 4.57. Understanding why is harder.

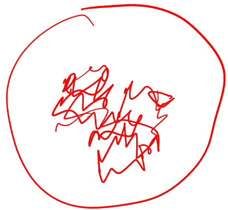
$\boxed{\text{Self fin} \iff \text{Disc wealth is a } \tilde{P} \text{ mg}}$

5. Stochastic Processes

5.1. Brownian motion.

$\mathrm{Var}(\xi_i) = 1.$

Pollen



- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\mathrm{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon}\, Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\mathrm{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

Weiner Process

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

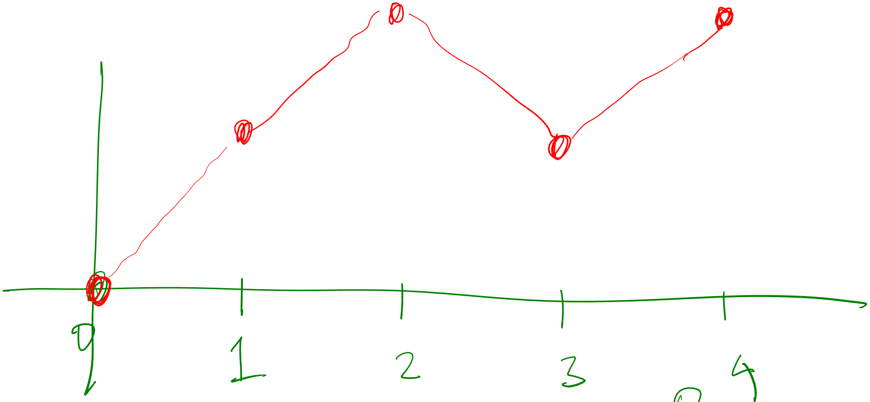
▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.

$X_0 = 0$ Outcome of k^{th} play is the RV $\boxed{\xi_k}$

Cumu wealth $X_1 = X_0 + \xi_1$

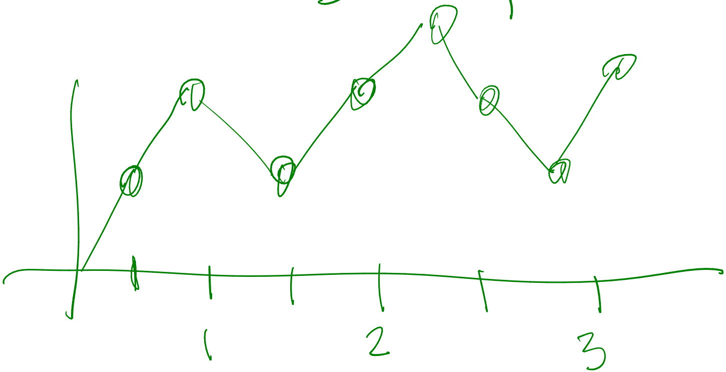
$X_{n+1} = X_n + \xi_{n+1}$



time

Speed up time

k,



$$\operatorname{Var}(X_n) = \operatorname{Var}\left(\sum_1^n \xi_k\right) = \sum_1^n \operatorname{Var}(\xi_k) \quad \text{(indep)}$$

$$= n.$$

$$\boxed{Y_t^\varepsilon = \sqrt{\varepsilon}\, Y_{t/\varepsilon}}$$

(Say t is an int mult of ε)

$$\operatorname{Var}(Y_t^\varepsilon) = \varepsilon \operatorname{Var}(Y_{t/\varepsilon}) = \cancel{\varepsilon}\ \frac{t}{\cancel{\varepsilon}}$$

- If t, s are multiples of ε: $\boxed{Y_t^\varepsilon - Y_s^\varepsilon} \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \boxed{\mathcal{N}(0, t-s)}$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

$\boxed{E\xi_i = 0, \quad E\xi_i^2 = 1}$

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

1

$\frac{1}{N} \sum_1^N \xi_i \xrightarrow{N \to \infty} 0 \quad (LLN)$

CLT $\frac{1}{\sqrt{N}} \sum_1^N \xi_i \xrightarrow{N \to \infty} N(0,1)$

Say t & s are mult of ε. ($s = m\varepsilon$ & $t = n\varepsilon$.)

Then $$Y^\varepsilon_t - Y^\varepsilon_s = \sqrt{\varepsilon} \sum_{m+1}^{n} \xi_k$$

$$= \sqrt{\varepsilon} \cdot \left(\text{sum of } \underbrace{\left(\frac{t-s}{\varepsilon}\right)}_{N} \text{ iid RV's} \right)$$

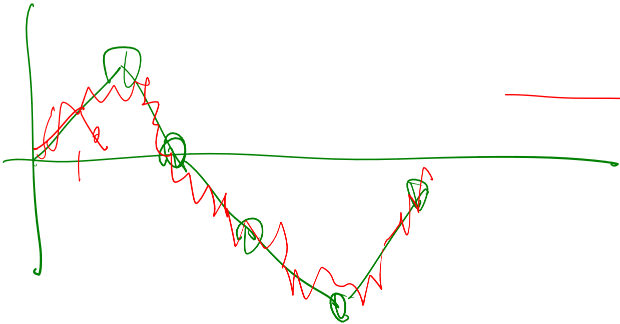
$$= \sqrt{t-s} \underbrace{\frac{1}{\sqrt{N}} \left(\text{sum of } N \text{ iid RV's} \right)}_{\to N(0,1)}$$

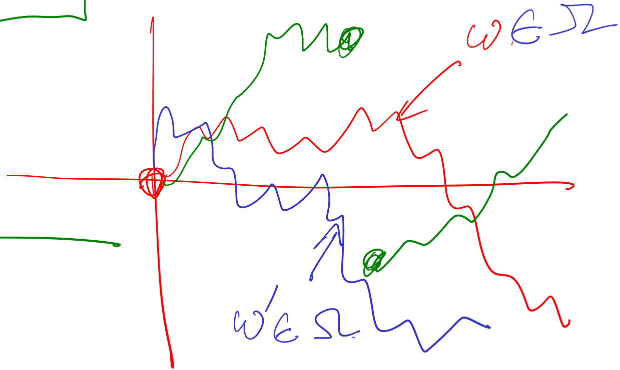
$$\left| \frac{t-s}{\varepsilon} = N \right.$$

$$\sim N(0, t-s).$$

5.2. **Sample space, measure, and filtration.**

- Discrete time: Sample space $\Omega = \{(\omega_1, \dots, \omega_N) \mid \omega_i$ represents the outcome of the i^{th} coin toss$\}$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.





- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} \colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\Big(\bigcup_1^\infty A_n\Big) = \lim_{n \to \infty} \boldsymbol{P}(A_n)$. $(\& \ A_n \subseteq A_{n+1})$
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

$X : \Omega \longrightarrow \mathbb{R}$

Given any $\omega \in \Omega$, $X(\omega) \in \mathbb{R}$.

Is $X > 0$?

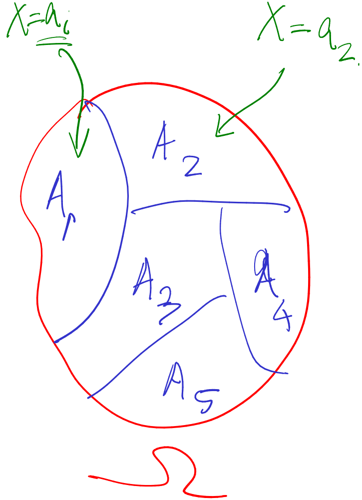
$\{X > 0\} = \{\omega \mid X(\omega) > 0\}$

Q: $P(X > 0)$

- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_\Omega X \, d\boldsymbol{P} = \int_\Omega X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$. $= \sum a_i P(X = a_i)$
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

Discrete rve

$$\boxed{EX = \sum_{x_i \in Ran(X)} x_i \, P(X = x_i)}$$



Proposition 5.3 (Useful properties of expectation).

α, β not random.

(1) (Linearity) $\alpha, \beta \in \mathbb{R}$, X, Y *random variables*, $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E} X + \beta \boldsymbol{E} Y$.

(2) (Positivity) *If* $X \geqslant 0$ *then* $\boldsymbol{E} X \geqslant 0$. *If* $X \geqslant 0$ *and* $\boldsymbol{E} X = 0$ *then* $X = 0$ *almost surely.*

(3) (Layer Cake) *If* $X \geqslant 0$, $\boldsymbol{E} X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt$.

(4) *More generally, if* φ *is increasing,* $\varphi(0) = 0$ *then* $\boldsymbol{E} \varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt$.

(5) (Unconscious Statistician Formula) *If PDF of* X *is* p, *then* $\boldsymbol{E} f(X) = \int_{-\infty}^{\infty} f(x) p(x)\, dx$.

lazy

($X = 0$ a.s. means $P(X=0) = 1$).

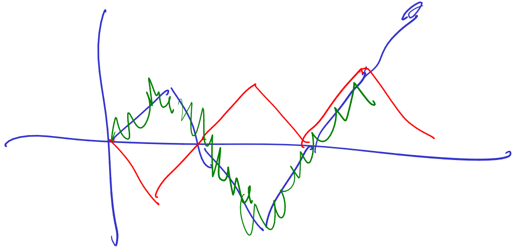
φ' = derivative of φ

PDF of X is p, then $E f(X) = \int_{\mathbb{R}} f(x)\, p(x)\, dx$

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\mathcal{F}_t$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0,1\}\}$.

$$X_{n+1} = X_n + \xi_{n+1}$$

(ξ_{n+1} = outcome of $n+1$th coin toss)



Eg: Say $s < t$.

$\{W_s > 0\} \in \mathcal{F}_t$

$\{W_s \in [1,2] \;\&\; W_t < 0\} \in \mathcal{F}_t$

$$\{W_{t+1} > 0\} \notin \mathcal{F}_t$$

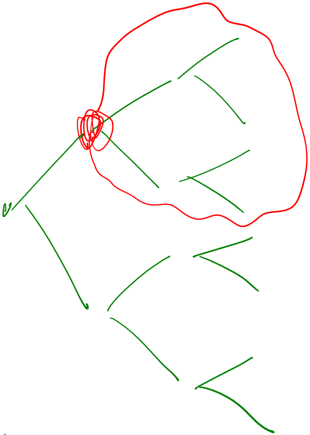
$$\{W_{t+1} > 0\} \in \mathcal{F}_{t+1}$$

5.3. **Conditional expectation.**

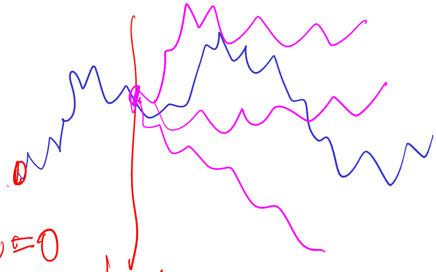
- Notation $\boxed{\boldsymbol{E}_t(X)} = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega'(s) = \omega(s)\ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

$E_1 X$

avg over all cointosses after the 1st one.



Dice



t=0

t=1

Cts

Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E}X$.

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.
(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.
(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*
(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X\boldsymbol{E}_t Y$.
(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).
(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E}Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

Lemma 5.8 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(\cancel{Y}), \qquad \textit{where} \qquad \cancel{g(y) = \boldsymbol{E} f(X, y)}.$$

Remark 5.9. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X, Y) = \int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$.

$g(x) = E f(x, Y)$

Set $x = X$

$g(X) =$ Avg $f(X, Y)$ in Y

& leave X alone

5.4. **Martingales.**

Definition 5.10. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.11. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Proposition 5.12. *Brownian motion is a martingale.*

Proof.